Optimal Control
Dr. Barjeev Tyagi
Department of Electrical Engineering
Indian Institute of Technology, Roorkee

Lecture - 15
Linear Quadratic Optimal Control Systems

Welcome to all. So, today our class is on linear quadratic optimal control system. In the previous class we completed our discussion on variational application, variational calculus application to the optimal control system and in this we have develop an approach in which in terms of the Hamiltonian we can represent our system where Hamiltonian is defined as v plus lambda prime f.

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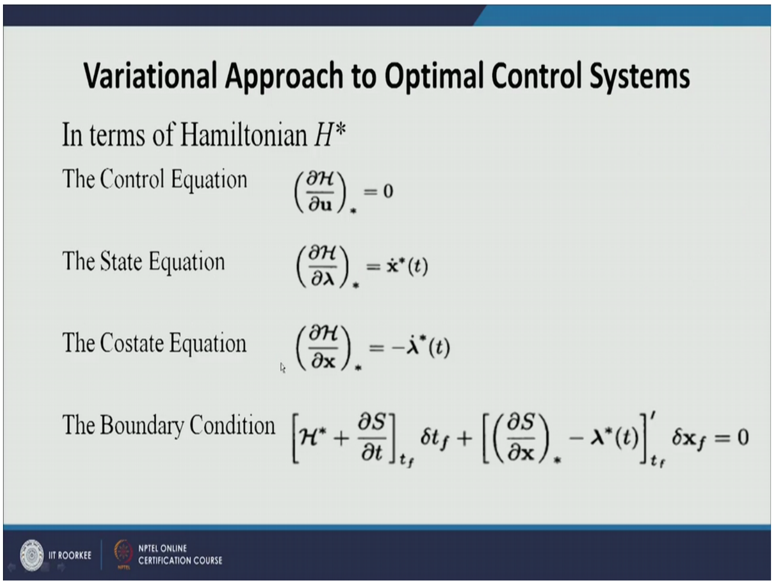

And our equations can be represented as the control equation del H by del u equal to 0, del H by del lambda equal to x dot which is my state equation, del H by del x equal to minus lambda dot which is my costate equation. These equation can be solved and the constant or the final solution can be determined utilizing the different boundary condition which comes up from the last equation.

So, now, application of this we will see to the linear quadratic optimal control system. As we all know what is a linear system.

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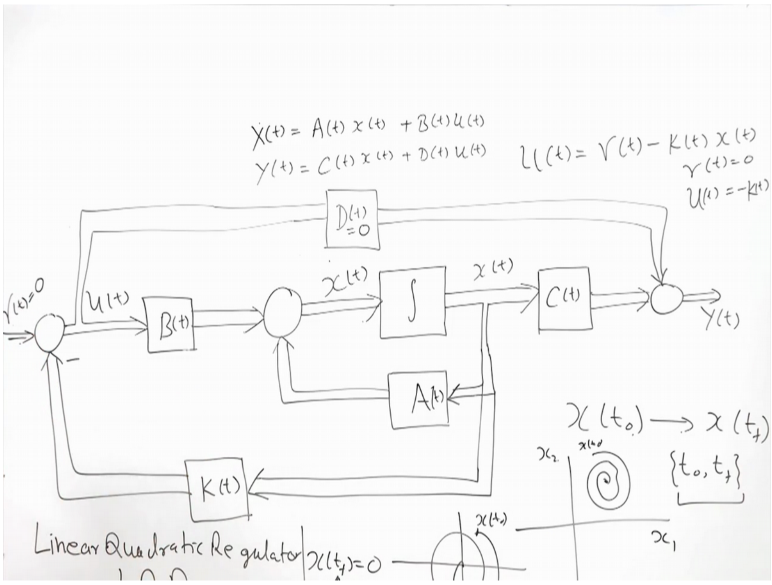

Any linear system in term in the study space we can represent it as x dot t equal to A of sorry, we take a general system which will be A t x t plus B t u t and point t we can represent as C t x t plus D t u t. So, objective here for the given linear system we have to develop a control low, control low are normally we take as u equal to sorry; we say this as u t equal to minus K t x t, so that we can control the system. And then the what condition we can control the system? If my system is completely controllable then I can control the system means I can transfer the state of the system from a given initial condition to some final condition in a given time, so that is my objective.

So, how to implement this control? Now this system I can represent in terms of the block diagram as sorry till this as B, this is integral, I take this as A and further if I will extent. So, this is my x dot of t, this is x t. So, x dot of t is A t x t plus B t, this I will take as u t and let us say this as my reference R t equal to 0 and C t x t will and as we have seeing if I will take this u I can pick up from here with D t, so this will be my y t. So, y t is C of x t plus D of u t.

So, this system I can represent in by this block diagram, objective here is to implement the control low u t equal to minus K t x t. So, in this case let first we consider R t is dot 0 then control low we can implement as feeding back the states as a negative feedback to r t. So, if I will consider this case let us first case u t equal to R t minus K t x t. So, this is a journal control, state feedback controller problem where we have to design the K t such that I can transfer my state from x t 0 to some x t f in a specified interval of time t 0 to t f. So, in this interval I want to transfer my state from initially state to final state for which we have to design a control low which way be R t minus K t x t. So, this is my control problem.

Now how to get the value of the K t there are the main approaches in advance control system we know we have the pole placement technique by which this K t can be determined. Now as if we will make R t to 0 what is the meaning this? My reference is 0, so if R t is 0 so my control low is nothing, but u t equal to minus K t take x t. So, this problem will become the regulator problem. So, in this we will, we do not have any change in the set point my set point is fixed, my system is subjected to the disturbance and with this disturbance system will again return to its operating point. So, that is called my regulator problem if R t will become 0.

So, I can also set let us say I have the 2 states x 1 x 2 and any operating point. So, let my system is start with x t 0 from here. So, after the disturbance it will return to the same point which is my operating point. So, this operating point if my R t is 0 I can take this operating point at the origin by simply using my shifting the origin principle. So, anytime I can shift my origin to this point. So, my problem here will be my from x t 0 I have to transfer my state to x t f equal to 0 where t f is specified. So, this is called my linear quadratic regulator problem which we call linear quadratic regulator, very famously we write this as this is my LQR problem.

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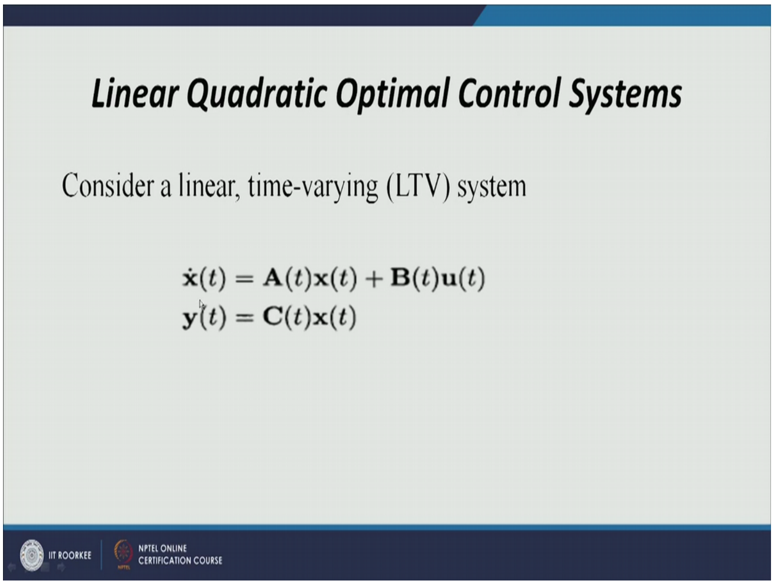

So, if you we will look at this we are defining a system as x dot t is A x t plus B u t y equal to C x t. In general we can consider the D matrix to be 0 this means if these person then you will have the direct influence on the y, in most of the physical system I have D equal to 0. So, we are representing our system simply as y equal to C x t. So, this is my linear time varying system because A B C all are the function of time we are varying with the time. So, initially we are considering a general linear time varying system objective is to keep the state x t near 0. So, this is a state regulator system.

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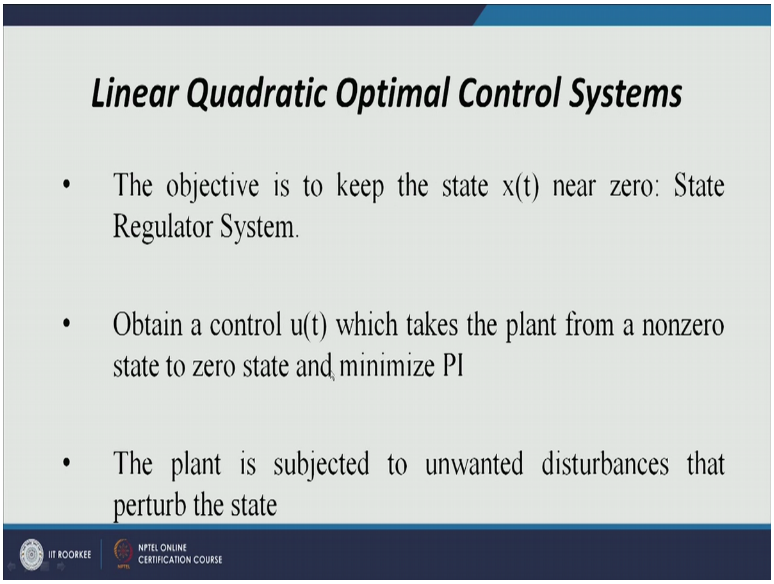

We want to obtain the control u t which takes the plant for nonzero is state to the 0 state and simultaneously minimize the performance index. In, as we are reaching transferring our nonzero state to the 0 state, so this means my R t is 0 and the plant is subjected to the unwanted disturbance that part of my system.

So, this regulator problem in this regulator problem our objective is to find the value of the K t and that is particularly optimally. If we have in a regulator u equal to K t x t as we said we can find this control low using many other approach one of the approach is the pole placement technique in which we specify where my closed loop system pole will lie. So, in LQR problem I will keep the system.

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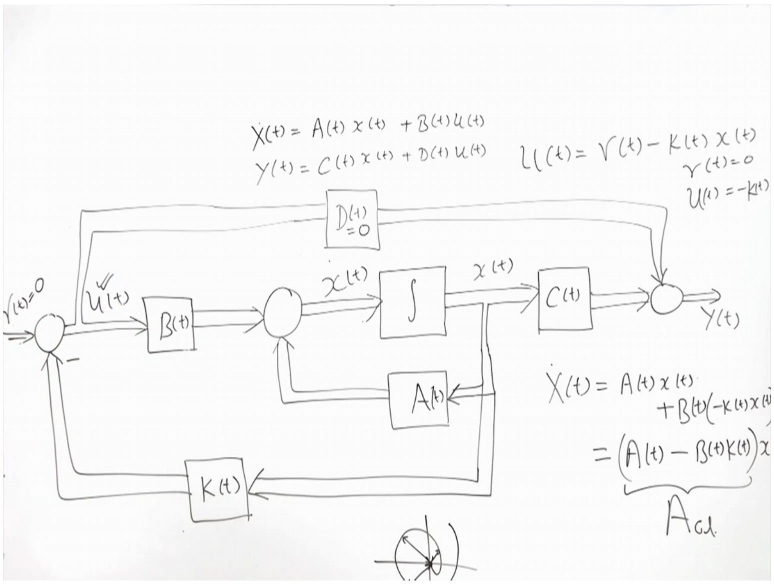

So, as u equal to minus K t x t if I will place this, so x dot t will be A t x t plus B t and in place of the u I am placing my minus K t x t so I feel combined the x term. So, my closed loop system will be A t minus B t K t and this whole is x t. And this whole I can represent as my close loop matrix.

So, I can say this A t is a open loop matrix and A c l is my closed loop matrix. So, after implementing the control low my objective here is whatever be the condition of the A t this A c l a closed loop matrix will be a stable matrix and have the desired performance characteristics, because in a linear system matrix A is my system matrix which completely governs the performance of my system. So, using the in a pole placement technique what we want, my closed loop system will have the desired poles which is already specified we are about to shift my eigenvalues of A in A c l means whatever be the eigenvalues of A in closed loop system I will shift my eigenvalues from, I will shift my eigenvalues to the desired location.

But in this case we cannot guarantee whether my eigenvalues are shifted optimally or not. So, that is why we have the state regulator problem to be solved optimally. So, that is why we are saying we our objective in linear quadratic optimal control system is to determine the u which will take the plant from nonzero to 0 state and simultaneously minimizes a performance index performance index.

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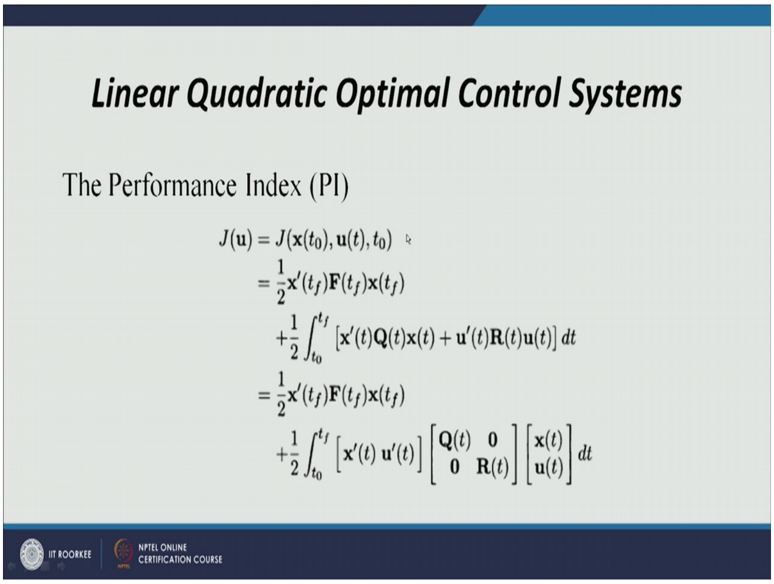

Here we take as 1 by 2 x prime t f, f of t f, x of t f, 1 by 2 integral t 0 to t f x prime q x plus u prime r u. So, because my performance index is quadratic in nature we call this problem a quadratic optimal control problem. So, if for a linear system this quadratic performance index we can select what this performance index will have, the first term one by 2 x prime t f F t f x t f represent my terminal cost which will depend on my what is my final state plus half of x prime q x. Now what actually is the x? In quality case if we consider my final state to it origin so this and this is my and initially state. So, at n instant of the time this is representing nothing, but the error in the state.

So, at n instant this is representing because my desire is 0 minus whatever is the state at that particular time. So, desired state minus the actual state giving me nothing, but the error and the desired state is 0 so this I am getting nothing, but error. So, the first part of integral performance index x prime Q x this nothing but used to minimize the error and second u prime R u is used to minimize the control effort which we have taken as the energy. So, this u prime R u is nothing but to minimize the control effort, minimize the error, minimize the terminal cost

So, we have selected the performance index which is quadratic in nature, which is minimizing by terminal cost, which is minimizing my error, which is minimizing my control efforts and matrix form we can write this as x prime u prime matrix Q and R, x and u. So, in this performance index we have taken as the F of t f, Q of t, R of t.

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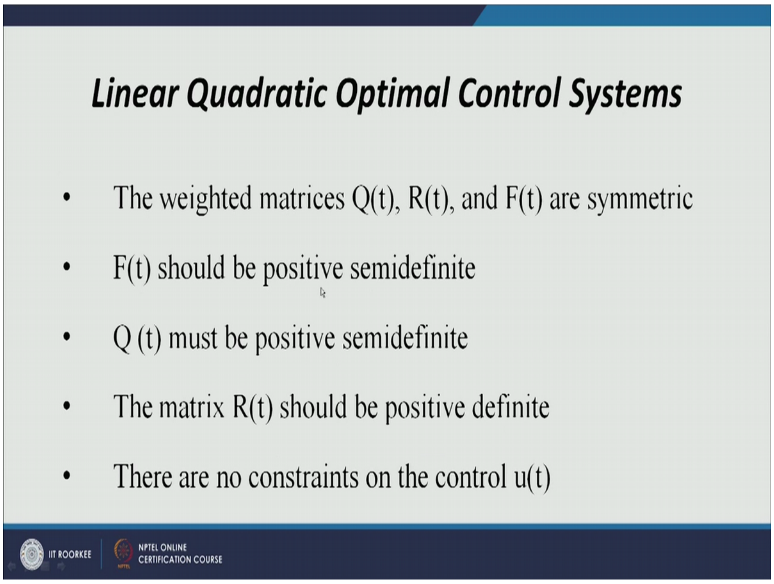

What these matrices represent here? These are the weighted matrix Q t R t and F t are they are symmetric in nature normally F t is positive semidefinite Q t is also positive semidefinite, but the matrix R t should be a positive definite. And there is no constraint are considered in this particular case. F is positive semidefinite means some of the state, weight given to some of the final state may be 0.

Similarly, weight on the state at any instant any state I can make it to 0, but anytime I cannot make any control to be 0 because all my system is controlled by the u t. So, always the certain rates I have to give to all my controls. So, that is why we are selected R to be the positive definite. So, we will take F positive semi definite, Q as positive semi definite, R to be the positive definite and we are not considering any constraint on the control. So, I have my plant which is given as x dot is A x plus B u, I have my performance index given as the J u and the this problem we have to and our objective is to find the u which will minimize the performance index as well as satisfy the plant condition.

So, if you will recall this was my variational approach to the control problem. So, we can directly use my Hamiltonian approach to solve this problem. So, we have to use equation, control equation, state equation and the costate equation to solve this linear quadratic optimal problem.

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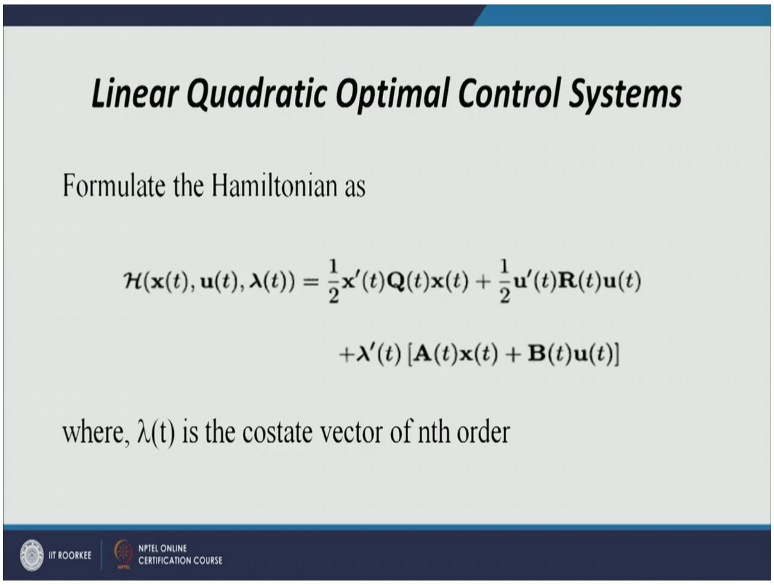

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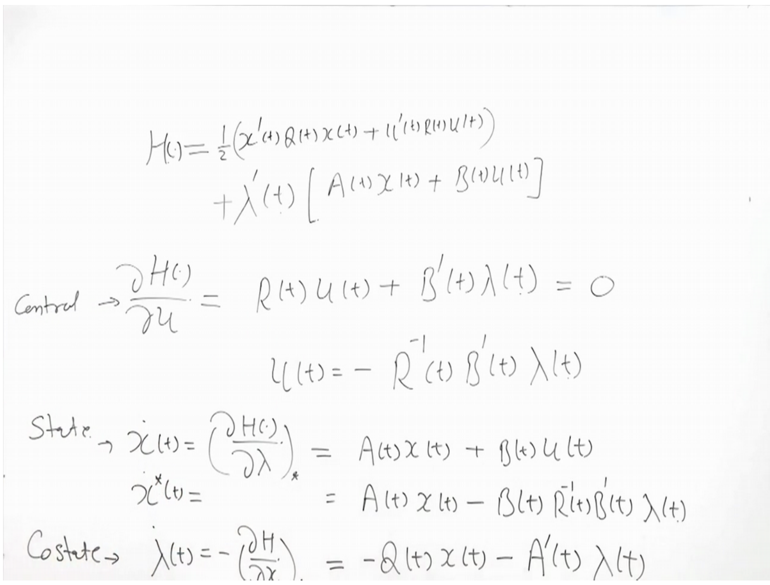

So, my first steps to write the Hamiltonian H which is equal to my v plus lambda prime f. So, what is my v? v is nothing, but my quadratic function which is x t Q t x t plus u prime R u and this whole we are dividing I taking as 1 by 2. This 1 by 2 is used to only to simplify the calculation which is come in the later part. This is my v plus lambda prime t into F. What is my f? Here x dot equal to A x this is A t x t plus B t u t.

So, I am taking this as my H which is half x prime Q x plus half u prime R u plus lambda prime A t x t plus B t u t. Lambda t is my Lagrangian multiplier, here we will take as a costate vector because my system is of the nth order therefore, the order of my lambda will also be of the nth. So, lambda is a nth order costate vector.

My first equation is del H by del u equal to 0. So, in the given H if I will differentiate this H with respect to u this is this time is a function of u and in the second my this lambda prime B u t is a function of u. So, their differentiation if the first time will give me 0 second because u prime R u this will be cancelled out so this will given me as R t u t and if I will differentiate this plus B transpose lambda t. So, differentiation of del H by u prime R u is R t u t and lambda prime t B t u t is B prime lambda and this must be equal to 0. So, therefore, I can directly write my u t to be minus R inverse B prime transpose lambda t. So, we got this equation has u equal to minus R inverse B prime lambda t.

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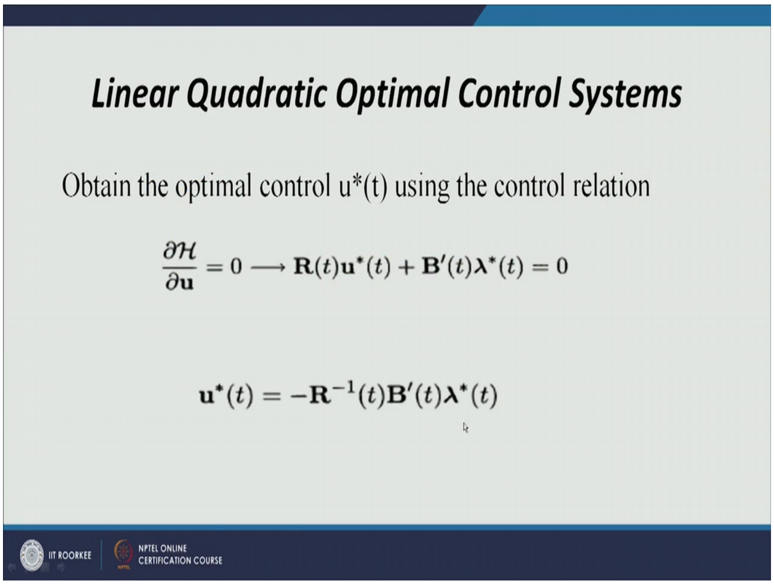

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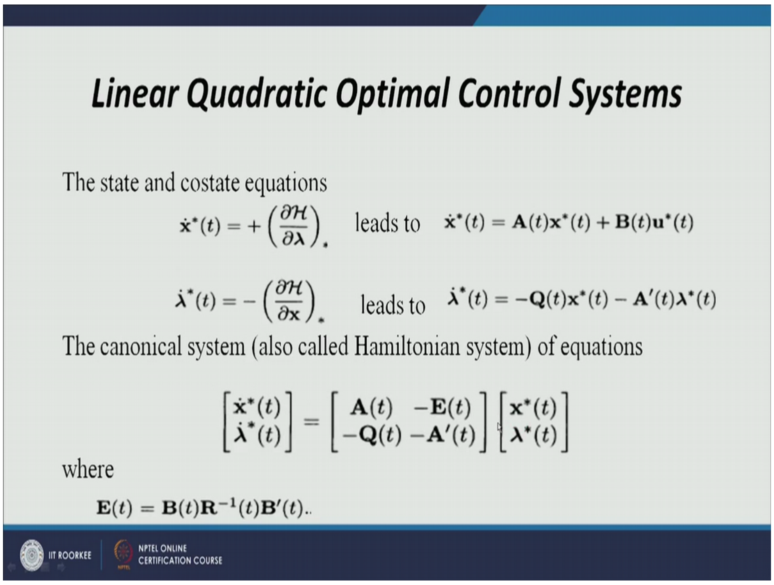

Then I have to write my state and the costate equation. State equation is x dot equal to del H by del lambda. So, del H by del lambda if we will take, so this is my control state equation is x dot t equal to del H by del lambda and what is the value of this.

So, the first term is independent of my lambda. So, this all will not give me anything and this will give me with respect to lambda. So, this is nothing but my A t x t plus B t u t I am getting and u is we already got R inverse this, so this will be my optimal value. So, this is nothing, but my A t. So, I can say this will be my optimally state as A t x t minus sorry; B t R inverse B prime lambda t. So, this will be my state equation costate equation lambda t equal to minus del H by del x is my nothing but lambda dot t.

So, my first term is a function of x. So, this is giving me due to this negative sign Q t x t and another x dependent term is lambda prime t A t x t I have to differentiate this term, if I will differentiate this term we will get minus A prime t lambda t. So, here prime remember represent my transpose. So, we are saying this is the A transpose and B transpose. So, I got the 2 equations as x dot is A t x t minus B t R inverse t B prime t lambda t and the costate equation is lambda dot minus Q t x t minus A prime t lambda t.

So, these 2 equations we are getting here and this equation I can write in this particular form which is called the Hamiltonian system.

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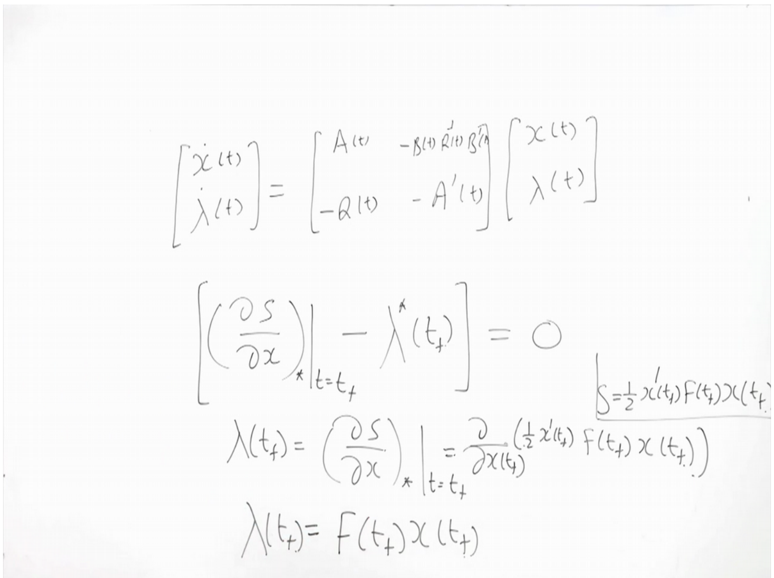

So, I am writing a state and the costate equation in terms of a matrix form some clubbing x dot of t lambda dot of t is x t lambda t. So, with the x t I will have the A t matrix A t x t minus B t R inverse t B prime t and second row is minus Q t minus A transpose t. So, I got this equation where t I am representing as B t R inverse B prime.

So, using this we have converted this into the Hamiltonian system. So, this is giving me the state and the costate equation which is converting my system into the Hamiltonian system.

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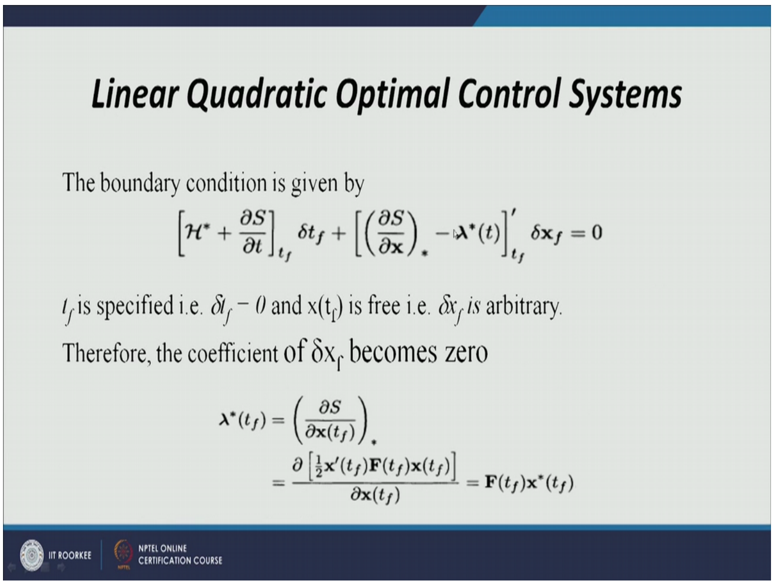

Next is my boundary condition here, the general boundary condition given as H plus del S by del t delta t f del S by del x minus lambda star t delta x f equal to 0. Say in this case if you will recall what is my problem, my problem is to transfer the state from initial state which is x t 0 means t 0 and x t 0 both are given to me to the final state in a specified time this will the t f is a specified. If t f is a specified, so this delta t f will be 0, so I have left only with the del S by del x minus lambda a star t at t f point delta x f. So, if I will consider my delta t f to be 0 I am left with the condition del S by del x at t equal to t f point minus lambda by star t say lambda by star t f at the t f point that must be equal to 0.

So, my lambda t f is nothing but del S by del x at t equal to t f point. Now if you will see what is my terminal cost, my terminal cost is half x of t f, f of t f sorry, x prime of t f, x of t f x of t f and this is nothing but my S, S equal to this. So, this means if I have to differentiate this with respect to x. So, this is nothing, but del x of t f of this my half x prime t f this is my S, F of t f x of t f this value I have to find and this is nothing but my F of t f, x of t f, this is giving me lambda of t f. So, by this we got my lambda t f of f of t f, x of t f. So, I am getting my lambda of t f of f of t f and x of t f. So, this is my terminal condition. So, what I will have here? I have my Hamiltonian system given as x dot t equal to A t minus B R inverse B prime minus Q minus A. So, this is my Hamiltonian system with final condition as lambda star t f equal to F of t f, x of t f.

So, today we stop here and in the next class we will complete our derivation to obtain the optimal control law for a linear quadratic optimal control system.

Thank you very much.