Optimal Control

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Lecture - 14
Variational Approach to Optimal Control Systems (Continued)

So, welcome friends to this class. This is again we are a still continuing the variational approach to optimal control system. In the previous class we have determine the Hamiltonian and the system in the Hamiltonian we can define by these 4 equations.

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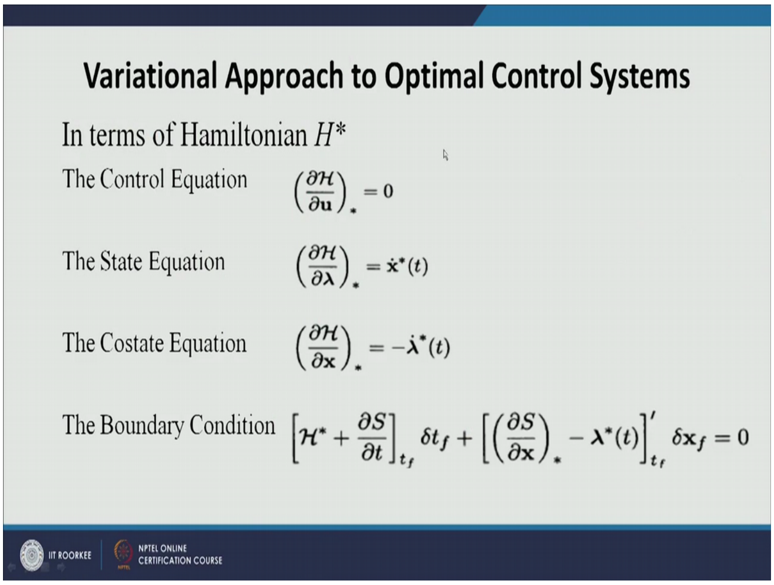

Last is my boundary condition, so in a given problem my first 3 equation always remain the same, but the final this last equation can change according to the condition of my end points. In all the case my initial points are given, but end points can vary problem to problem if we will take the different systems my first system is the fixed final time and the fixed final state system. We already have been discussed before.

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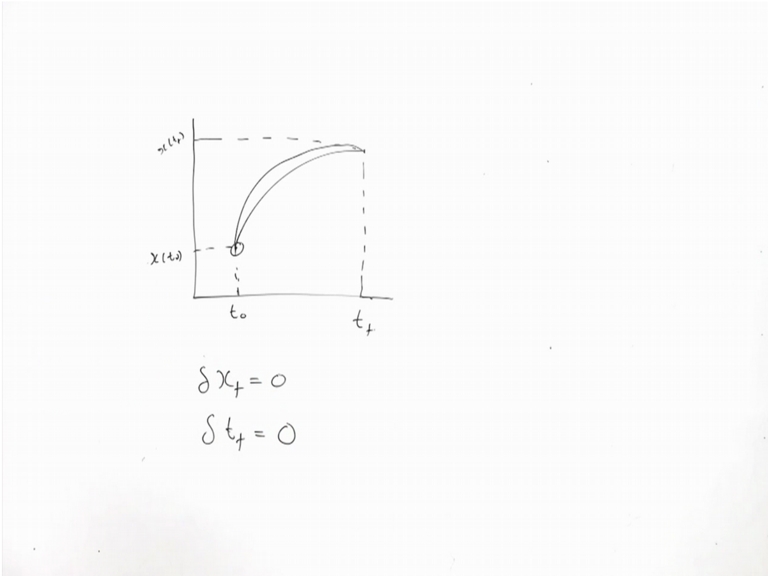

So, I am given with t 0, x t 0 this normally is given. This is my t f point if I will fix up the end point at x t f. So, if variation will take place subjected to the fixed initial point and the fixed final point initial point we already are considering for all the cases to be fixed, but as in first case if we will fix up the end point. So, what will be the condition my delta x f will be 0 my delta t f will also be 0.

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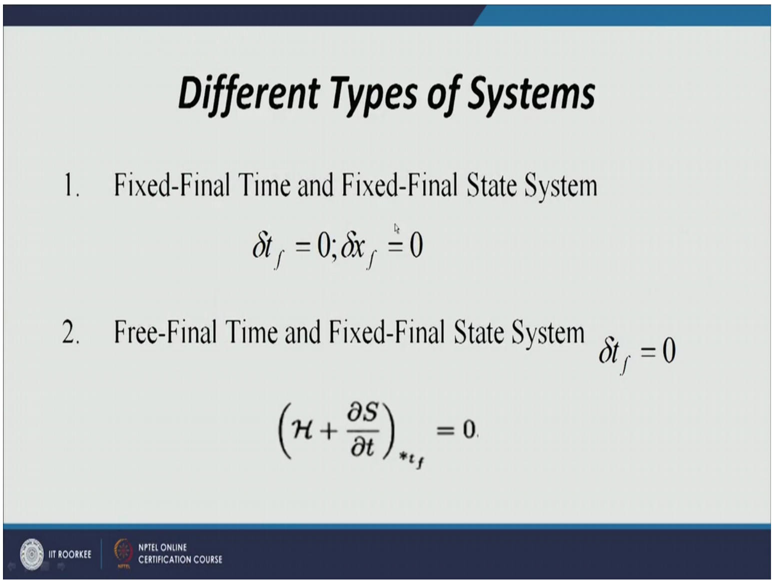

So, these 2 delta t f and delta x f will be 0, this means delta t f is 0 delta x f is 0, so last equation will not appear in the problem. But t f and the x t f they already are given that is why they are fixed. So, we are with the final condition.

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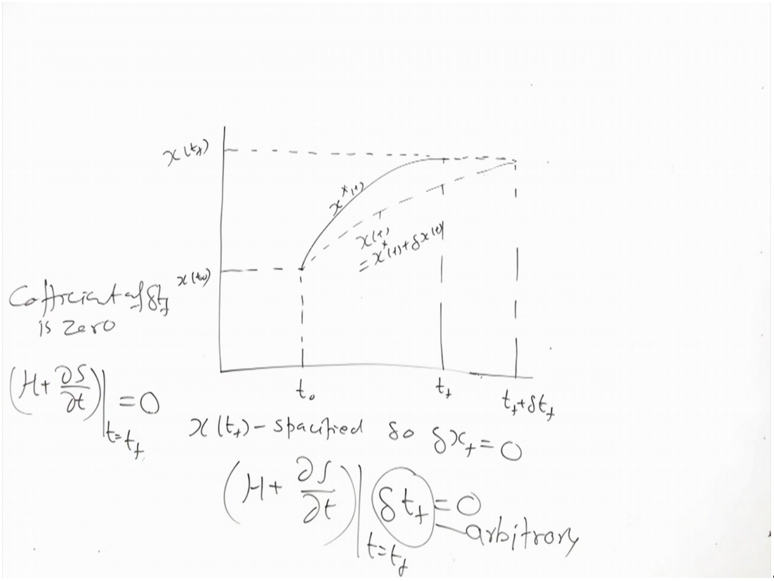

So, this I know the initial condition I know the final condition based on this I can solve my first 3 equations. My second cases the free final time, but the fixed final state the system is free final time means my final state is fixed. So, I am given with the t 0, x of t 0 my x t f is fixed and t f is free. So, if this is x star of t, this is t f and let say this is t f plus delta t f. So, I can say this is my x t which is nothing but x star of t plus delta x of t. So, that is my second case which is the free final time, final time is free and the fixed final state where x t of is free.

So, my trajectory can land anywhere this we are x t f remain the fixed. So, in this case if we will see to my final condition as x t f is specified. So, what I will have? I have delta x f to be 0. So, delta x f will be 0, but, so H plus del S by del t. So, what is left? H plus del S by del t t equal to t f point, delta t f that must be equal to 0 and this delta t f is arbitrary. So, delta t f is arbitrary. So, the coefficient of delta t f is 0, this means H plus del S by del t at t equal to t f point that will be equal to 0. So, we will have H plus del S by del t at t f point that will be 0.

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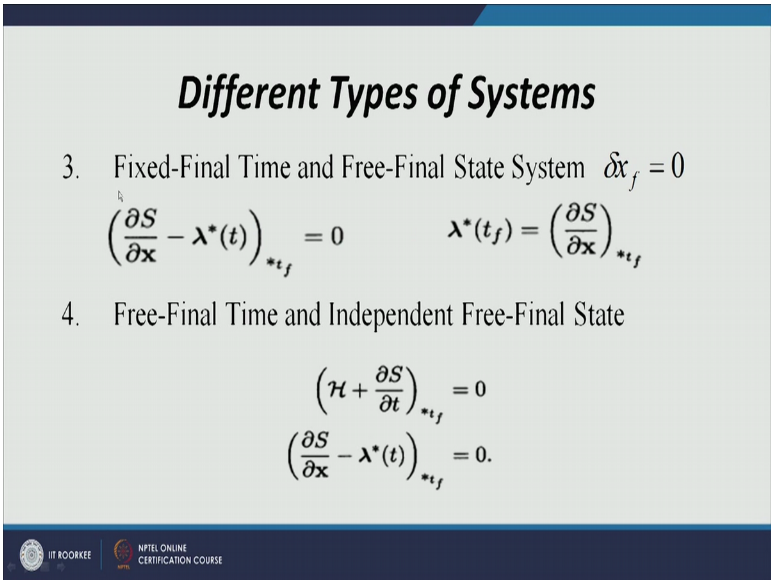

So, this is my second case the third case naturally if we will have the fixed final time and free final state system.

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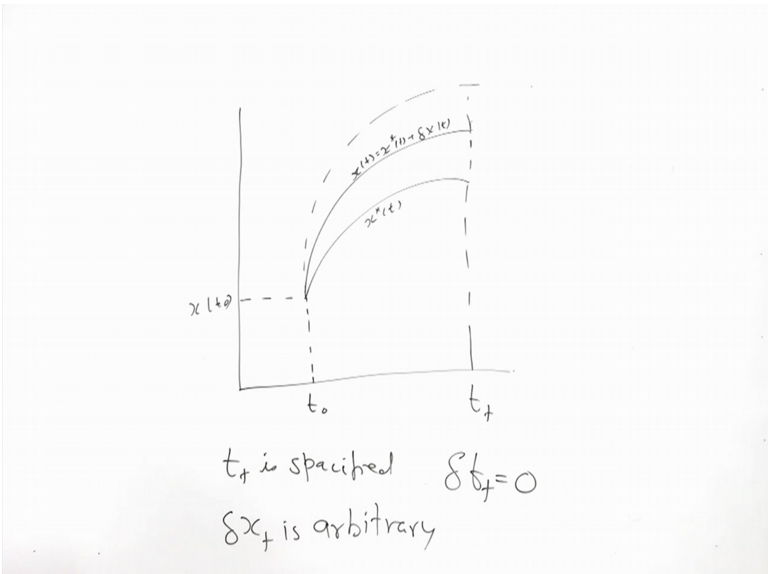

So, we have the free final state and fixed time. Again we start with t 0, x t 0 and this t f is specified, t f is. So, t f will be fixed if this is my x star t, so my trajectory can go anywhere. So, this will be my x t which is x star t plus delta x t. So, on the t f line my trajectory can terminate. So, in this case naturally my f t f is specified, delta t f will be 0 and the coefficient has delta x f is arbitrary delta x f is arbitrary. So, the coefficient of delta x f is del S by del x minus lambda prime t equal to 0.

So, this coefficient 0 this means I have a condition lambda t f equal to del S by del x at t f point. So, this will be my boundary condition in that case.

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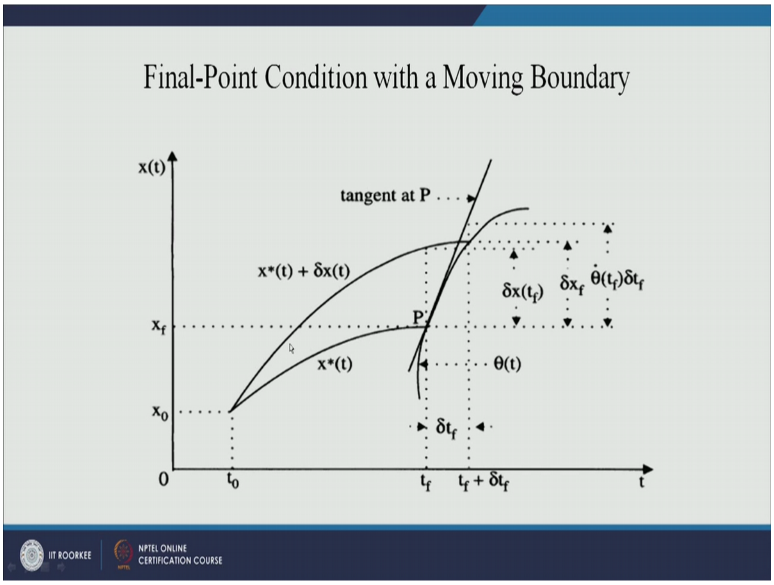

If free final time and the free final state free, so t 2 t f plus delta t f and free final sate. So, my trajectory can go anywhere. Again the 2 cases adjust here which we once we have the free final time and the free final state if both are independent means there is no relation between delta t f and delta x f. So, the coefficient of both will independently will be 0. So, H plus del S by del t will be 0, del S by del x minus lambda star t at t f point that will also be 0.

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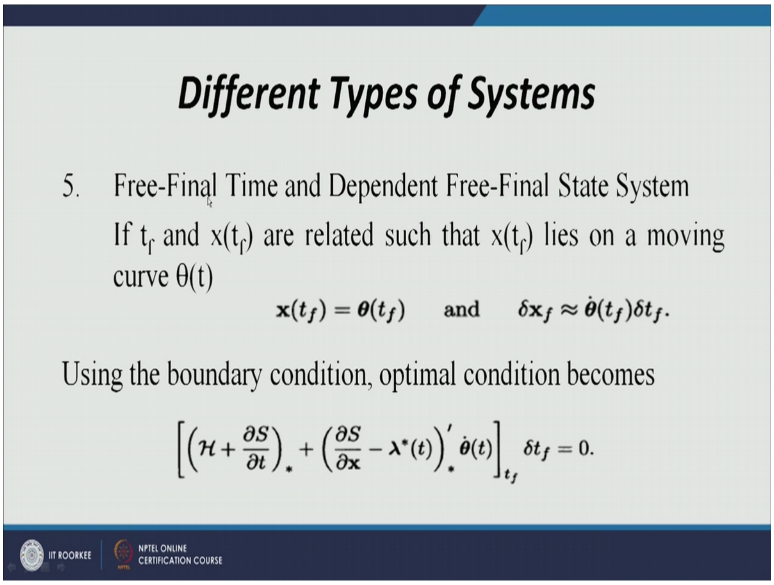

But there may be the case that free final time and the dependent free final state system, as we have considered before let us say my trajectory terminate to a curve given by the theta t. So, this means we are considering a case, can I have the slide; we are considering the case that my final state is terminating to a curve theta t which is given by this curve.

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So, at t f point I have x of t f equal to theta of t f. So, at this point my x t f is equal to theta of t f. So, in this case I can approximate delta x f is theta dot of t f into delta t f. So, if I am all approximate this, so I have a relation between delta x f and delta t f. I will place this value delta x f S theta dot delta t f in this given relation. So, here t f and delta x f are related to each other I am replacing delta x f as theta dot of t f and if I will club this my terminal condition comes out to be H plus del S by del t plus del S by del x minus lambda star t theta dot delta t f equal to 0.

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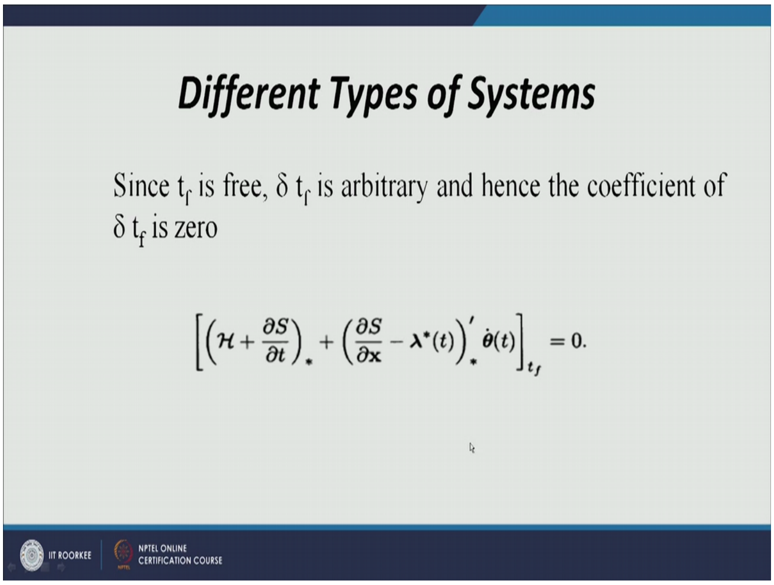

Delta t f is arbitrary, so the coefficient of this will be equal to 0 to get my final condition. So, in total we have discussed the 5 type of the system - the first one was the fixed final time and the fixed final state in which delta t f and delta x f both are 0, my second was the free final time and the fixed final state I have the fixed final is state, so delta x f will be 0. My third case is the fixed final time and free final state system. So, delta t f will be 0 and the condition you will get it here. The fourth and the fifth case we have the free final state and the free time, but in the first case both are independent. So, the coefficient of both delta t f and the delta x f are independently 0 and if it is dependent by a curve say theta t then my condition will be H plus del S by del t del S by del x minus lambda t prime theta t equal to 0.

So, these end point conditions can be utilized if we are solving a problem. So, next we will take up an example and see how we can apply the Hamiltonian approach to solve a optimal control problem.

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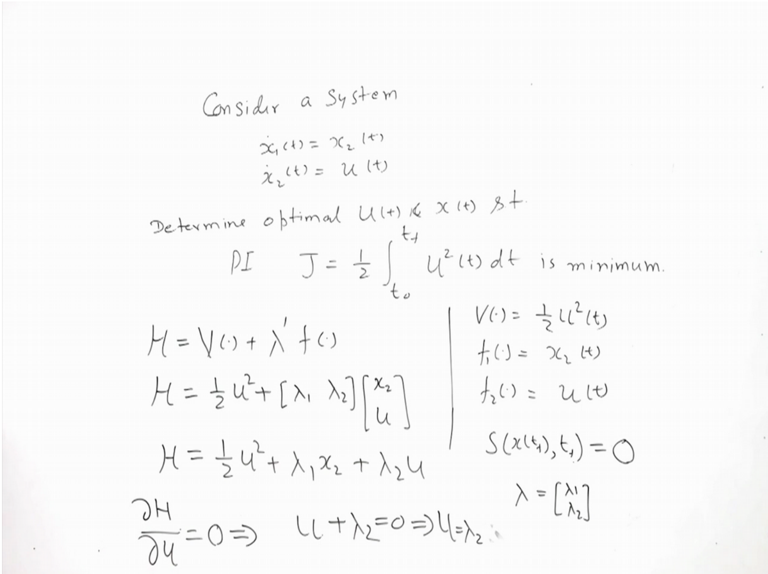

Let first we will define; so we are given with consider a plant which is given as x 1 dot t S x 2 t and x 2 dot t S u t. So, our objective is determine optimal u t and x t such that performance index J equal to t 0 to t f, u square t d t is minimum. So, our simple problem we are considering a simple plant x 1 t equal to x 2 and x 2 dot equal to u objective first a optimal u we will find, with the optimal u we say our is straight will also be the optimal. So, we are objective is to find the optimal u and optimal x which will minimize the J.

So, this problem we will solve using the Hamiltonian approach should define the Hamiltonian we have V plus lambda prime f. So, in this case if we will see what we have V equal to integrant here half of u square t f 1 is x 2 t, f 2 is u t and if we will see the terminal cost S x t f, t f as no cost is given, so this value will be 0. So, based on this value I can define my H which is nothing but my half u square. So, I am dropping the t plus lambda prime, what is the lambda prime? It is a Lagrangian multiplier for each condition we will have the one Lagrangian multiplier. So, this is nothing but my lambda 1 lambda 2 or we can say I am defining my lambda S, lambda 1, lambda 2.

So, my lambda prime is lambda 1 lambda 2; f 1 f 2 which is nothing but my x 2 u. So, I can define my H as half u square plus lambda 1 x 2 plus lambda 2 u. So, here this H is defined as this equation. What is my objective to solve this? Using this 4 equation. So, my control equation is del H by del u equal to 0. So, first I will find out the control. So, first step I take del H by del u equal to 0 if I will differentiate H with respect to u. So, my first step u is here. So, this is nothing but u, second term independent of u, third term is plus lambda 2 equal to 0. So, this will give me nothing but u equal to lambda 2. So, t both the side we are dropping out.

So, we can say u equal to lambda 2 with the given value of the H now we write the other equations. So, I keep my; this equation with me. So, my second equation is my state equation which is giving del H by del lambda equal to x dot t.

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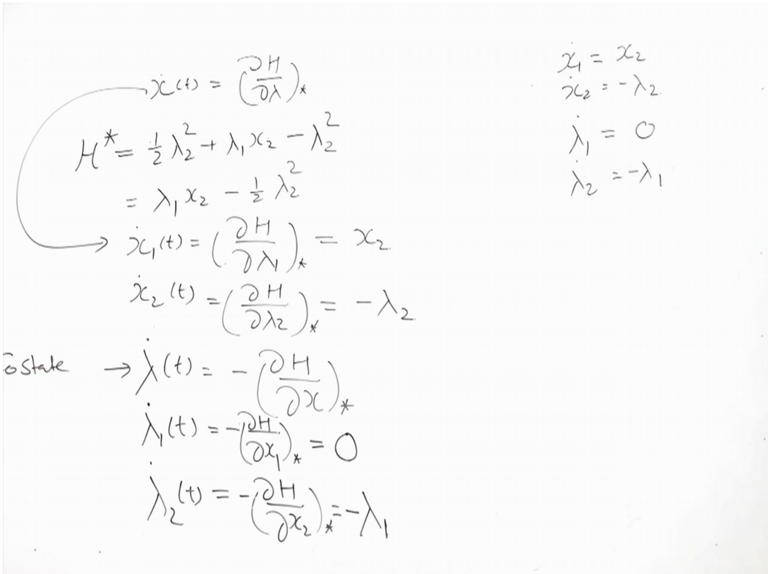

So, x dot t we are taking as del H by del lambda at optimal point, optimal point means H we are finding at the optimal. So, u is sorry; in this case my u is not lambda 2, but this is u equal to minus lambda 2. If I will replace my u in this, so I can write my H star is 1 by 2 u, the optimal value of the u is minus lambda 2, so minus, this is nothing but half of lambda square plus lambda 1 x 2 minus lambda 2 into u. So, this is minus lambda square. So, my optimal H is nothing but lambda 1 x 2 sorry; minus 1 by 2 lambda 2 is square.

So, on this what is my optimal a state x? So, x 1 dot t is del H by del lambda 1 if I will take this del H by del lambda 1 because x is a vector of x 1 and x 2. So, this is del H by del lambda 1 is my x 1 del H by del lambda 2 is my x 2. So, del H by del lambda 1 if I will take, so this is nothing but x 2 and x 2 dot of t is del H by del lambda 2 at optimal point and this is in this case minus lambda 2.

So, my x 1 dot is x 2 x 2 dot is minus lambda 2 similarly now we can go. So, now, we write the costate equation is lambda dot t S minus del H by del x. So, my costate equation is lambda dot t as minus del H by del x for optimal value. So, what we have to do. So, this my optimal H is this, so del H by del x. So, again lambda is a vector of lambda 1 lambda 2. So, we will write lambda 1 dot t as minus del H by del x 1, lambda 1 del H by del x 1. With respect to x 1 if we will differentiate this what I will get there is no x 1 term, so this is 0. Similarly we have lambda 2 dot t minus del H by del x 2, what we will get by this? With respect to x 2 we are differentiating. So, we have only the lambda 2. So, this is, so lambda 2 t is nothing but lambda 1.

So, from the a state and the costate equation we will have the state equation is x 1 dot as x 2 and x 2 dot as minus lambda 2 which we are getting by these 2 equations, and the costate equation lambda 1 dot equal to 0 and lambda 2 do equal to lambda 1 sorry; we have did a mistake del H by del x 2 is this, this is lambda 1 with negative sign. So, this will be minus lambda 1. So, my this equation is nothing, but x 2 equal to minus lambda 1 because we have miss this negative term initial to represent this. So, now, these equations we have to solve simultaneously, how we can solve this equation?

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$$\dot{\lambda}_1 = 0 \Rightarrow \lambda_1^*(t) = C_1 \rightarrow (1) \qquad \dot{x}_1 = x_2$$

$$\dot{\lambda}_2 = -C_1 \Rightarrow \lambda_2^*(t) = -C_1 t + C_2 \rightarrow (2) \qquad \dot{x}_2 = -\lambda_2$$

$$\dot{x}_2 = C_1 t - C_2 \Rightarrow x_2^*(t) = \frac{1}{2} C_1 t^2 - C_2 t + C_3 \rightarrow (3) \qquad \dot{\lambda}_1 = 0$$

$$\dot{\lambda}_2 = -\lambda_1$$

$$\dot{x}_1 = \frac{1}{2} C_1 t^2 - C_2 t + C_3 \Rightarrow x_1^*(t) = \frac{1}{6} C_1 t^3 - \frac{1}{2} C_2 t^2 + C_3 t + C_4 \rightarrow (4)$$

Let us consider the boundary conditions are given as

$$x(0) = [x_1(0) \quad x_2(0)]' = [1 \quad 2]'; \quad x_1(2) = 0$$

$x_1(0) = 1$ from Eqn (4) $\quad x_1(0) = 1 = C_4 \Rightarrow C_4 = 1 \quad$ $x_2(2)$ is free

$x_2(0) = 2$ from Eqn (3) $\quad 2 = C_3 \Rightarrow C_3 = 2 \quad$ t_f is specified

$x_1(2) = 0$ from Eqn (4) $\quad 0 = \frac{8}{6} C_1 - 2C_2 + 5 \quad$ $x_2(t_f)$ is free

So, we take first equation lambda 1 dot equal to 0, so solution of this as we now, this solution gives us lambda 1 t equal to let say C 1 where C 1 will be a constant if lambda 1 t is C 1 we have lambda 2 dot is minus lambda 1 which is nothing, but C 1. So, this means lambda 2 t is minus C 1 t plus C 2. So, this will be my lambda 2 t. Say from these equations x 1 dot is x 2, x 2 dot is minus lambda 2. So, what we can write? Next equation we can take as x 2 which is minus lambda 2 means this is C 1 t minus C 2. So, this is x 2 dot if I will simplify this I will get x 2 t as, so 1 by 2 C 1 t square minus C 2 t plus C 3. So, this will be my x 2 dot, and if I will have next equation is x 1 dot as x 2 which is 1 by 2 C 1 t square minus C 2 t plus C 3. So, this will give me my x 1 t S just to integrate this 1 by 6 C 1 t cube minus 1 by 2 C 2 t square plus C 3 t plus C 4. So, I have lambda 1 t, lambda 2 t, x 2 t, x 1 t and we can note this these all are my nothing, but optimal values because they are determined on the optimal u.

So, next is these equations these set of equations we got simply by solving the first 3 equations. Now to determine the value of all the constant C 1 C 2 C 3 C 4 we have to utilize the boundary conditions, and boundary condition will depend what type of the case we will have. Like you will consider one case here also though these conditions can vary, let us consider the boundary conditions are given as, let we are given with x 0 which is nothing but my x 1 0, x 2 0 transpose is equal to 1 2 the another condition is given as let x 1 2 equal to 0 and x 2 2 is free. So, I am given with these conditions like x 1 0 equal to 1 I have x 1 t given here. So, what this equation give me? So, let us gives the name equation 1 equation 2 let say this is 3 and this is 4.

So, x 1 0 equal to 1; we can have from equation 4 what we will have? x 1 0 equal to 1 and this equal to t equal to 0 nothing, but my C 4 this implies C 4 equal to 1. Now x 2 0 equal to 2 we can take the equation 3, from equation 3 we have because x 2 0 is 2 equal to and this case t is 0. So, this all will be 0 we are left with the C 3. So, C 3 equal to 2. So, we are evaluated the 2 constants C 3 and C 4. Now my another condition here is x 1 2 equal to 0 means t f is fixed and see for x 1 again a fixed condition is given, x 1 2 equal to 0.

So, again from 4 - x 1 2 is 0. So, again we consider the equation 4 what actually we will have? My t equal to 2, so 0 equal to I am taken the t S 2, C 1 is unknown. So, this is nothing but 8 by 6 C 1 minus t equal to 2 4. So, 2 C 2 t equal to 2 C 3 is 2. So, 4 plus C 4 is 1 plus 5. So, I get a equation is 8 by 6 even minus 2 C 2 plus 5.

Again I need one more equation because I cannot find C 1 C 2 from this equation. So, I will use this which is given x 2 2 is free, this means t f S is specified and x 2 t f is free.

So, which condition I will use from my case? My condition is I have the fixed final time and the free final state and that is applicable to my second state for x 2 only, my condition is lambda t f equal to del S by del x t f.

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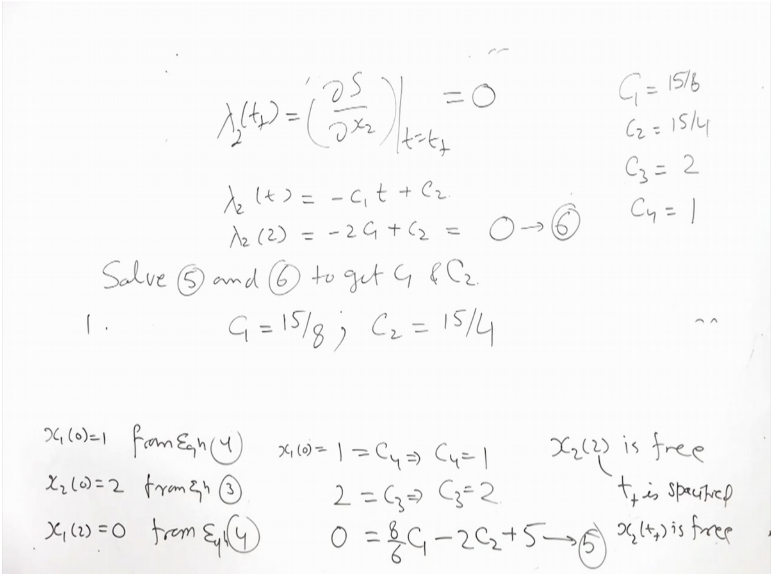

So, I can write lambda t f S del S by. So, lambda 2 t f del S by del x 2 at t equal to t f point as you can see we have considered the S to be 0. So, this is nothing but this will gives me equal to 0. So, my lambda 2 t f is 0 this means my I will have equation number 2 which will have lambda 2 t is minus C 1 t plus C 2. So, t f is 2, so I am writing lambda 2 2 which is my t f minus 2 C 1 plus C 2 and lambda 2 value is nothing but 0. So, 4 equation we have, so I have this equation number 5 this equation number 6. So, equation 5 and 6 can be solve 2 find the value of C 1 and C 2 which gives me, so solve 5 and 6 to get C 1 and C 2.

So, what actually we are getting? So, this gives me C 1 equal to 15 by 8 and C 2 equal to 15 by 4. So, now, I have all my C 1, C 2, C 3, C 4 as C 1 equal to 15 by 8, C 2 equal to 15 by 4, C 3 equal to 2 and C for equal to 1. So, with this C 1, C 2, C 3, C 4 are calculated utilizing the endpoint condition. And now I can represent my all optimal values in terms of the C 1, C 2, C 3 and C 4.

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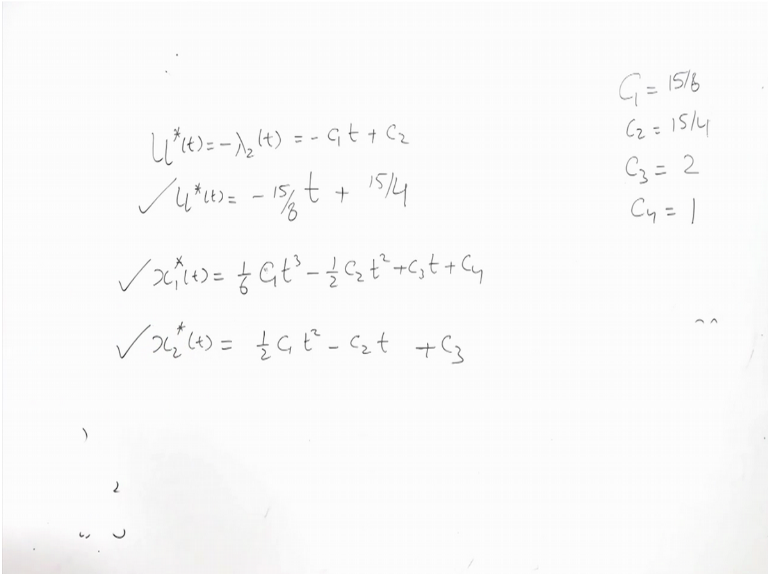

So, what I will have? My optimal u t was, so u star t was minus lambda 2 t and lambda 2 t is nothing but my minus C 1 t plus C 2. So, my optimal u is nothing but minus 15 by 8 t plus 15 by 4, and x 1 t is given as 1 by 6 C 1 t cube minus 1 by 2 C 2 t square plus C 3 t plus C 4. So, I can place this value to get the optimal x 1 and similarly optimal x 2 is given as 1 by 2 C 1 t square minus C 2 t plus C 3. So, this is my optimal control relation and these are the optimal a states. So, this example gives the application of the Hamiltonian approach to determine the optimal control and the optimal states of a given system.

Now, depending upon the final conditions my values of C 1, C 2, C 3, C 4 can change. So, different boundary condition can be taken as we have discussed the 4 cases we will end of with the different type of the conditions. Like the case 1 2 3 4 these equations can be utilized to determine the coefficient. So, if I will summarize my approach I have to solve first the control equation to get the optimal control then get the state and the costate equation, solve these equation with optimal control value in terms of the C 1, C 2, C 3, C 4 or whatever be the constants are there and these constants can be determined utilizing the initial condition t 0 and x t 0 as well as the final conditions which may appear in this 5 phones.

So, I stop my; this lecture here and in the next class we will further see the application of the Hamiltonian approach to determine the journal case of a linear control system.

Thank you very much.