Optimal Control
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Lecture – 12
Variational Approach to Optimal Control Systems

Welcome friends to this class, in this class we will discuss the variational approach to optimal control system. In the previous classes we have seen that how we can determine the optimum value of a function of a functional without condition and with condition. So, this variational approach to optimal control system that problem we can take it as the optimum value of a functional with some given condition. So, in this case our condition will be nothing but my plant model.

So, we start our discussion with a journal plant model, and then in the later stage we will discuss how we can implement this theory to the linear control system. So, first we will define our problem.

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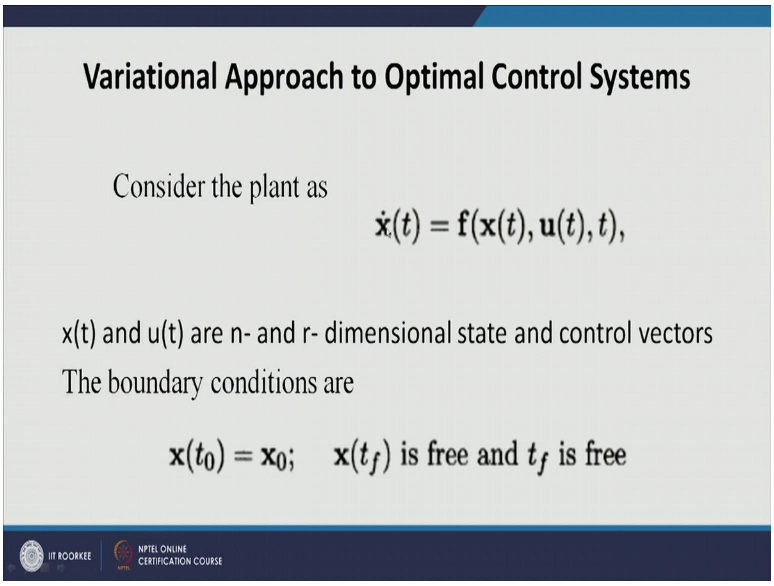

We are given with a plant which is a time varying plant given as the x dot t equal to f of x t u t and t. Now here this f may be linear or non-linear function and x and u are the n and the r dimensional state and the control vectors. So, this means it is a multivariable means the states are from x 1 to x n, and u is varying from u 1 to u r. My boundary conditions are x t 0 equal to x 0 means the initial condition is defined it is specified; t 0 is given x 0 is given, but my t f and x t f are both are free.

So, it is a free endpoint problem.

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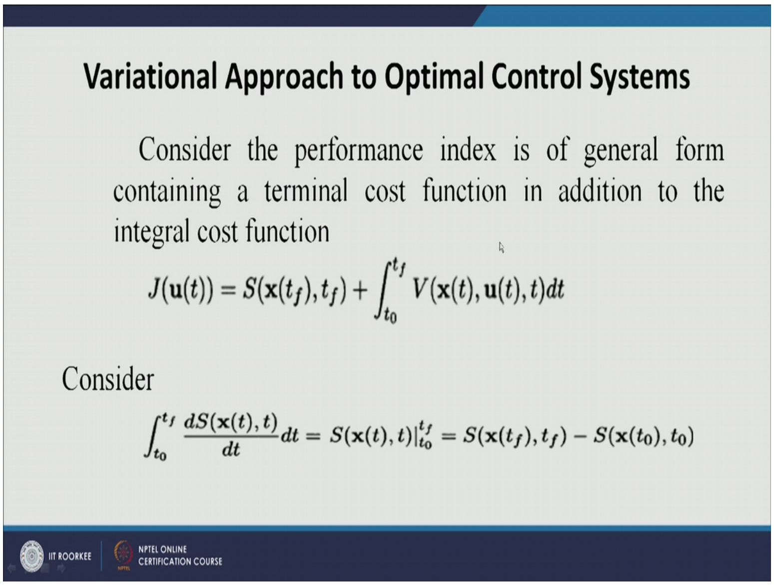

So, objective here is we have to find out the optimal value of the u, optimal value of the x such that it will minimize the given performance index. What is the performance index here it general form of the performance index is considered which will have the terminal cost given as s x of t f t f, and the integrant part which is nothing but t 0 to t f V x u and V as a function of x u and t dt. So, we have to find out the optimal u which will minimize the performance index subjected to the boundary condition given here. So, this is my problem, this problem means we can take the minimization of the j, this is my functional, the plant which is given as x dot equal to f x u t this I can take it as my condition as in the previous lecture we have taken as the g.

So, this I can write in the equality form this I can use it as my performance index and so this is a problem of the minimization of a functional with many variables subjected to my plant condition x dot equal to f x u t. How to solve this? Let us consider the integral of d s by dt term from t 0 to t f. So, we are integrating d s by dt multiplied with the dt. So, dt dt will be cancelled out, you are only with the d of s x t t. So, I can write this is s x t t from t 0 to t f means value of the x at t 0 point, value of the s at t f point which can simply be written as s x of t f t f minus x of x t 0 t 0. In this case if you will see my t 0 and the x t 0 points are given, basically this is a constant term. So, x of t 0 t 0 is a constant term.

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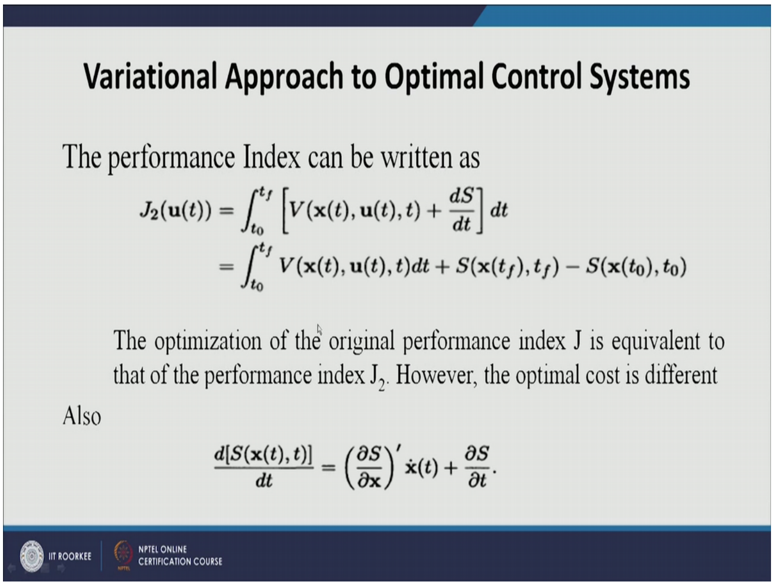

Now what we are doing, we are modifying my performance index as j 2 by adding this d s by dt taking into the integrated.

So, I am considering my performance index as V plus d s by dt dt; if I will expand this I will keep this V with the integral from t 0 to t f into dt, and d s by dt I will expand using this criteria. So, this give me nothing but s x of t f t f minus s x t 0 t 0. So, this I can include in. So, this two will give me nothing but my j. So, j 2 is nothing but your j minus s, x t 0 t 0 this s of x t 0 t 0 is a constant. So, what actually the minimization of the j or the minimization of the j 2 is a same thing, only difference here is the value of the j will be different than the value of j 2, to find the j 2 from j we are subtracting s of this which is a constant. So, if I will find out the j 2, I can place the value of the; I can at the value of s into the j 2 to get the value of the j.

So, my original performance index j is equivalent to nothing but the performance index of j 2. However, in both the case my optimal cost will be different. Now one more step we will see before coming to the actual problem d by dt of terminal cost, d by dt of s of x t t what we can write here del s by del x prime into x dot because d by dt of x we are taking plus del x by del t. So, further this d s by dt can be expanded by this.

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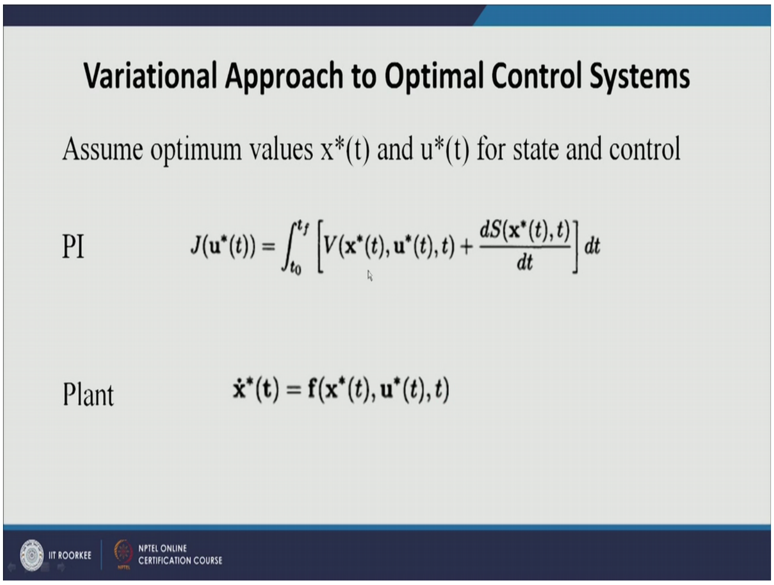

This means I can suppose my problem is my objective is to minimization of the j which is nothing but I have which I have considered as the j 2, and my plant condition is x dot equal to f x u t. Now if I will consider let x will have the optimal value as the x star, which is given as this curve in this figure. So, this is my x star.

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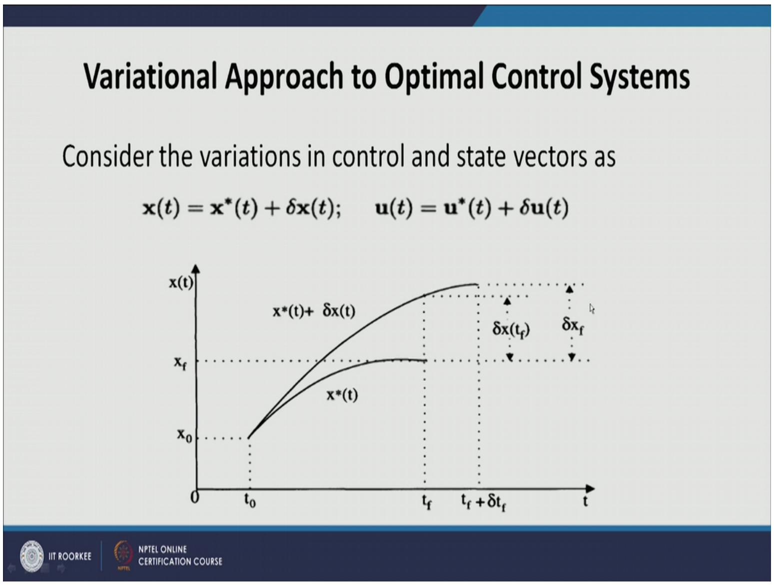

Let of this x star will have the variation is delta x. So, x star delta x will be my variational is trajectory. So, this is my optimal trajectory which I am saying as the x star, this is my variational trajectory which is x star plus delta x. All other these term where it is required then at that particular time we will discuss this, so at optimal point I can define my j as j u of a star which is nothing but V at the optimal point x star u star and d s by dt at x star this, and my plant will become now because the optimal I have assume at optimal point this is given as nothing but f of x star u star t.

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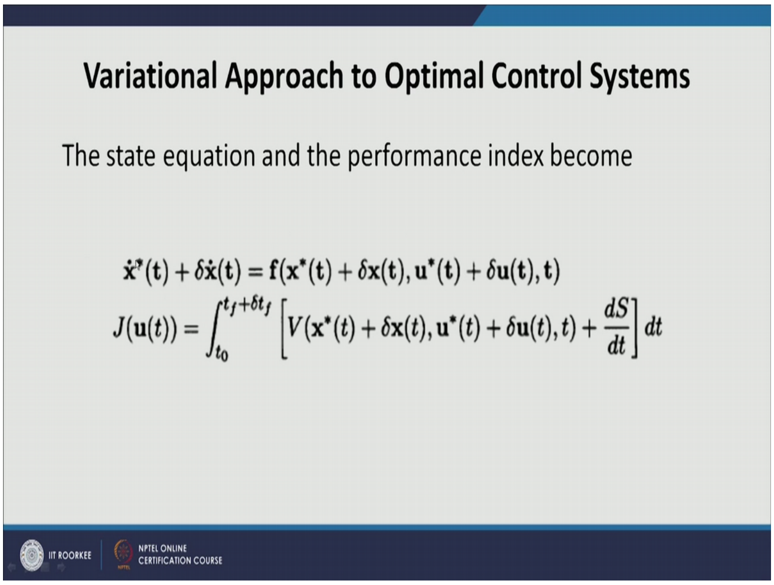

Now we are giving a variation at the optimal point as x plus delta x. So, at variation point my plant will be what, my plant was the x dot equal to f x u t here is the variation of the delta x. So, I can write x dot plus delta x dot t as f of x t plus delta x, u t plus delta u t and t.

Similarly, I can write the j at the variational point x plus delta x, which is written as j u. So, if you will see this I am writing as a j of u star means this is a performance index at the optimal point this I am taking as the, j u of t at the variation point.

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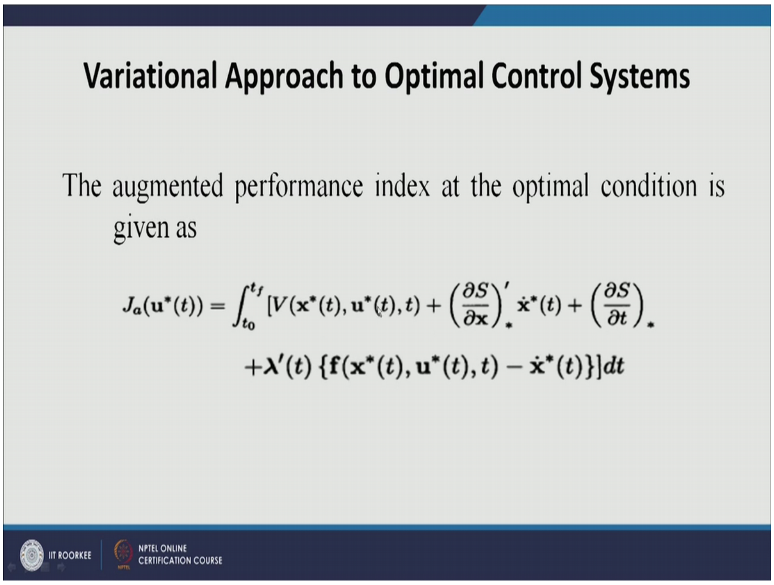

Now we are writing the augmented performance index, as you can recall in a we have done in the previous class to write down the augmented functional or augmented performance index which is nothing but my V plus lambda prime g. So, in this case what is my v? V is the whole which we are considering in the integrant. So, we have considered the optimal value of the x and the optimal value of the u, and at these optimal value we are defining your performance index as j u star which is nothing but V plus d s by dt dt and the plant at the optimal point is x star f x star u star t.

So, this is my optimal trajectory x star, and we are giving a variation of delta x into this to define my another trajectory as my boundary point is free. So, my t f is free from t f to delta t f, and x t f is free from x of t f to x of t f plus delta t f. So, my endpoint is free so this can go to anywhere. So, we have defined the j at the optimal point similarly I can define the j at my variation point. Similarly, I can also define my plant at the variation point which will be x star plus delta x dot equal to f x plus delta x u plus delta u and j will be nothing but V of x plus delta x, u plus delta u plus d s by dt dt. So, we have defined j at optimal point and at the variation point. So, now, we have the p i s j and plant has x dot.

Now, I can write my augmented function as j a, a for represent for the augmentation. So, this augmented function for optimal condition, this means the V plus lambda prime g will be my Lagrangian, now what actually is my v? V we have taken as.

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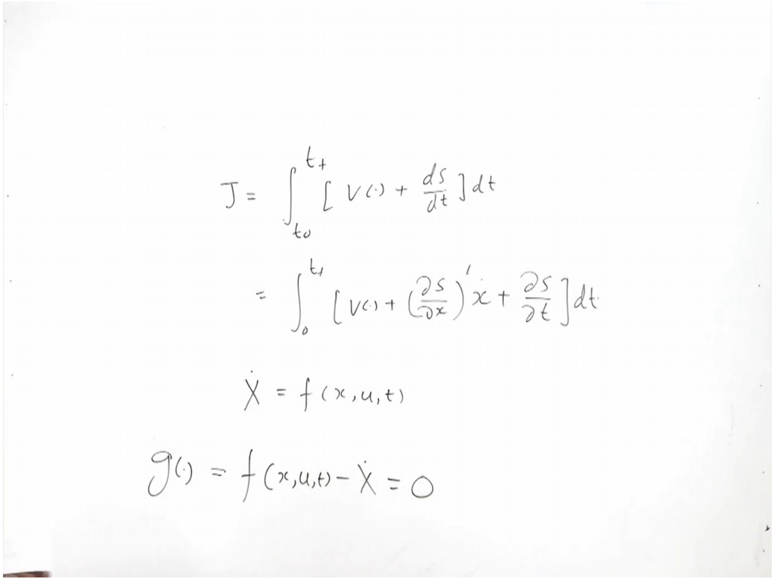

So, we have taken j as integral t 0 to t f, V plus d s by dt dt; and what is the d s by dt which we have taken here. So, you can recall this we have written as the del s by del x prime x dot. So, this we can write as t 0 to t f, V plus d s by dt is del s by del x prime, x dot plus del s by del t dt.

So, substituting d s by dt as this value n at optimal point for j u star and at variation point in j u; so first I am writing my j u star as V plus del s by del x x dot star plus del s by del t plus lambda prime, what is my g? I am taking my plant as x dot equal to f x u t, t I have dropped from here just for the simplicity. So, I can write my f x u t minus x dot equal to 0 and this is nothing but my g. So, augmented function how we write? V plus my augmented function is this whole which is my V in this case plus lambda prime g.

So, at optimal point we are writing a augmented function as V plus del s by del x prime, x dot plus d s by dt plus lambda prime f x u t minus x dot t in to dt; and if you will see this is nothing but my Lagrangian function V plus lambda prime this is my g if you will compare it with my previous class a simple optimal value of a functional with condition. So, my complete functional is this integrant lambda prime plus sorry lambda prime into my condition g.

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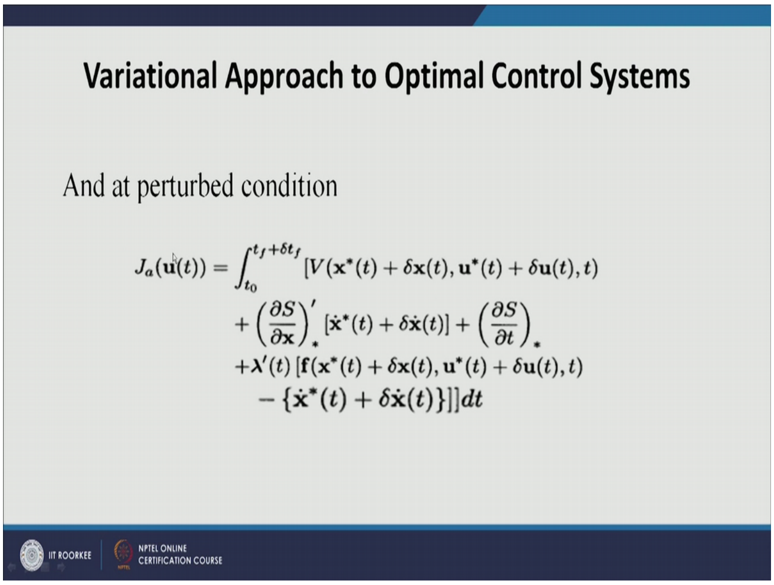

Similarly we can define this augmented function at variation point where my V will be x plus delta x, u plus delta u t, del s by del x now because there is a variation in the x. So, this is x dot plus delta x dot, d s by dt plus lambda prime and my condition will be what here, f function x plus x t point u plus u t point sorry delta u t point minus x dot plus delta x dot dt.

So, I have my j augmented function at optimal point, augmented function at variation point. So, this is nothing but my Lagrangian here define my Lagrangian at the optimum point my Lagrangian at the variation point.

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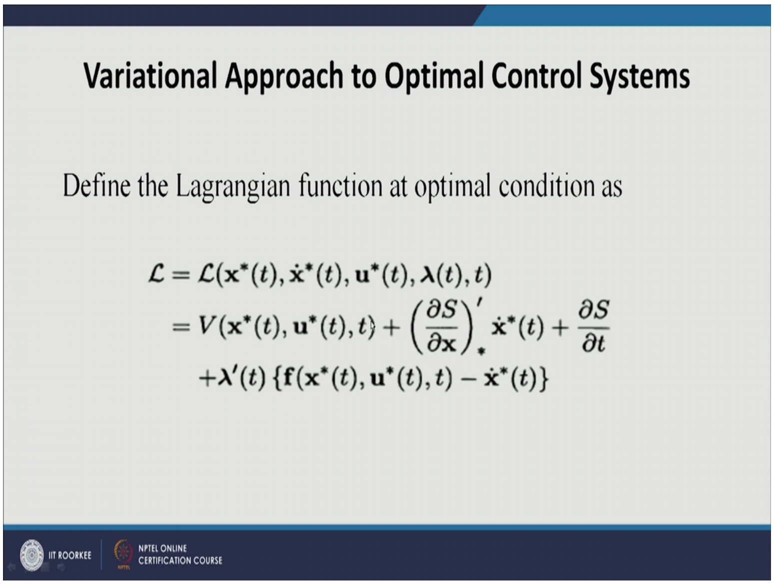

So, this at optimum point Lagrangian I am writing as L which is nothing but V plus del S by del X prime x dot del S by del t plus lambda prime this. So, this is nothing but my functional integrant which we have taken plus lambda prime and this is my condition and you can note this, this we are writing at the optimal point.

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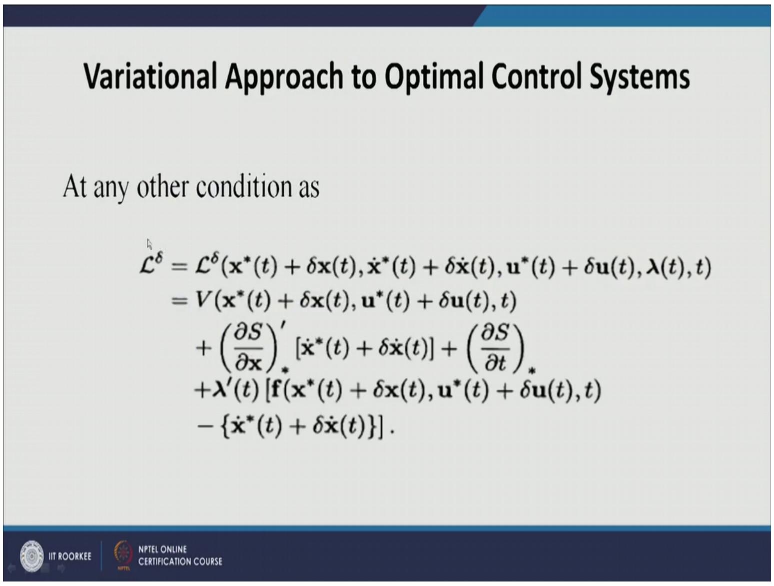

Next L delta we are defining at the variation point, where L delta is a function of x plus delta x, x dot plus delta x dot u plus delta u lambda and t; which can be written as V at the variation point del s by del x prime at variation at the x dot plus lambda prime f at the variation point, minus x dot at the variation point.

So, we are defining my L at the optimal point L is what this is my Lagrangian, Lagrangian at variation point is L delta. So, this is J a at variation point is nothing but my L delta times dt.

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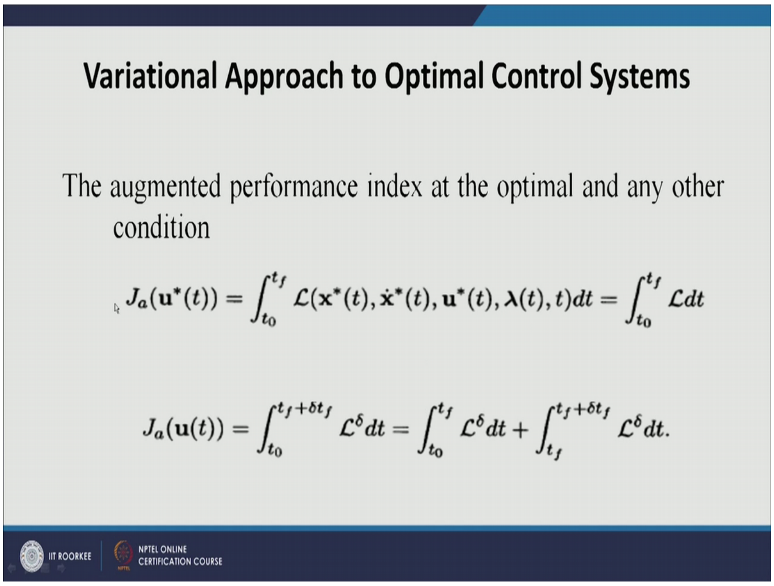

So, at optimal point what is my J? This is t 0 to t f, L dt and what is the augmented function at the variation point because my x t f and t f both are free. So, my limits are from t 0 to t f plus delta t f; as we are varying at variation point my limits will change from t 0 to t f plus delta t f; why as we can see here because my at variation point this is terminating at t f plus delta t f point. While in optimal trajectory we are taking up to the t f point. So, that is why we are saying my augmented function at optimal point from t 0 to t f augmented function at variation point from t 0 to t f plus delta t f, in terms of the Lagrangian I can define my augmented function as t 0 to t f L of dt here L already we have defined previously, the value of the j from t 0 to t f plus delta t f of L lambda dt.

So, this t 0 to t f plus delta t f L lambda dt we can break up into the two integrals; one is from t 0 to t f another is from t f to t f plus delta t f. So, this integral t 0 to t f plus delta t f has been broken into the two integrals, one is from t 0 to t f other is from t f to t f plus delta t f. Now here we will consider the second term of J a u t which is nothing but.

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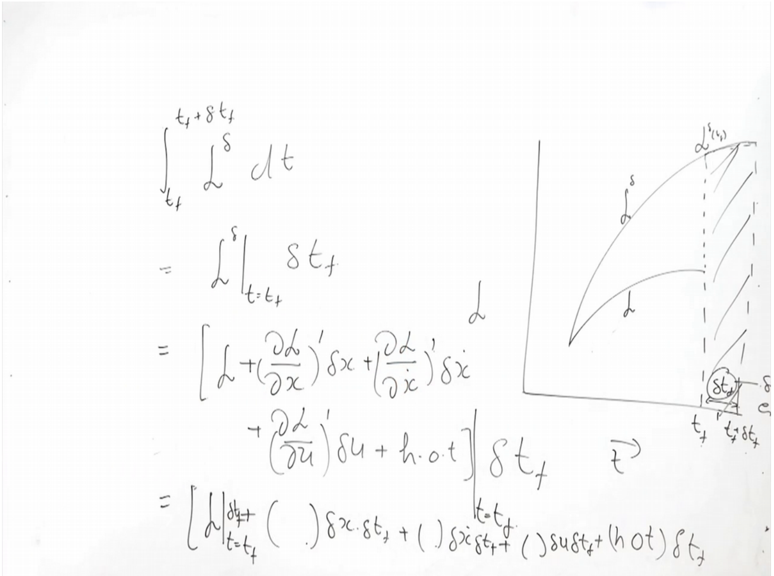

So, we are considering t f to t f plus delta t f L delta dt. Now if you will see what is my this is my t f point, this is t f plus delta t f. So, if I say this is my L and this is my L lambda with respect to time this I am taking as my L which is varying.

So, from t f to t f plus delta t f we were evaluating this integral. So, this is L lambda at the t f point. So, I can approximate this integral nothing but the area under this curve because this is multiplied because this is my delta t f and this is the value which can be integrated if t f is considered to be the small enough sorry this delta t f is considered to be the small enough.

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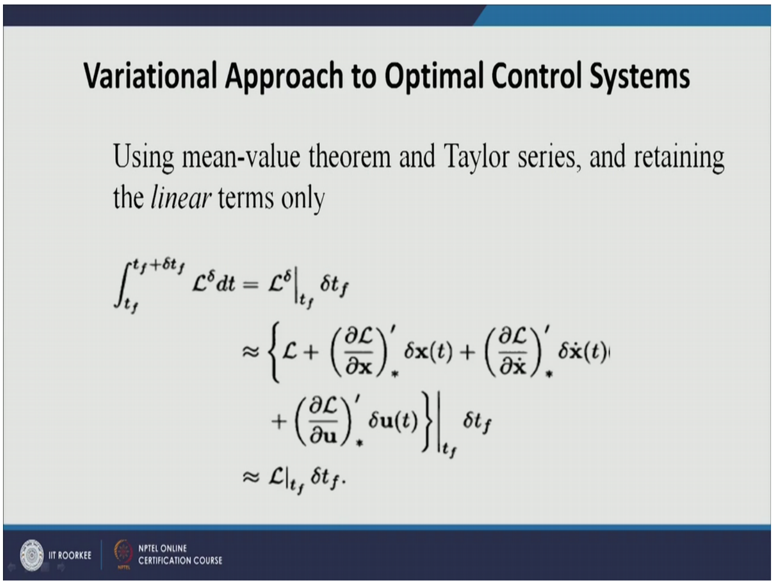

So, we can take this integral as nothing but the area under this curve and can be written as L lambda at the t f point multiplied with delta t f; this means this whole area we are considering as my integral and this integral is giving me nothing but L of this L delta at t f point multiplied with delta of t f.

So, this we are writing this integral evaluated at equal to t f point multiplied with delta t f. What is this L delta? This is the Lagrangian function at the variation point, this variation point I can explain using my Taylor series and variation is where this variation is at the optimum point. So, I can approximate this as L where the Lagrangian at the optimal point plus del L by L is a function of x. So, this is nothing but del L by del x, x is a vector, so we can take this prime as delta x this is a function of x dot. So, del L by del x dot prime delta x dot, this is also the function of u. So, del L by del u delta u we also have the higher order terms this whole multiplied with sorry.

So, we already have taken the bracket here. So, this L delta I am expanding in this way, this is L Lagrangian at the optimal point the first order derivatives then higher order derivative. So, all derivatives with respect to x, x dot and u for which the L is a function and this is multiplied with delta t f, and naturally because this whole is evaluated it equal to t f point. So, now, if I will multiply this I get the terms like L at t equal to t f point plus a term with delta x into delta t f plus a term with delta x dot delta t f plus another term delta u delta t f plus higher order terms multiplied with delta t f.

So, accept the first term sorry this is also multiplied with delta t f. So, accept the first term my all other terms are the second order term. So, these can be neglected and my L t f I can approximate directly in terms of the L at the t f point into delta t f; where L t f point is the optimal value of the L at t f point delta t f.

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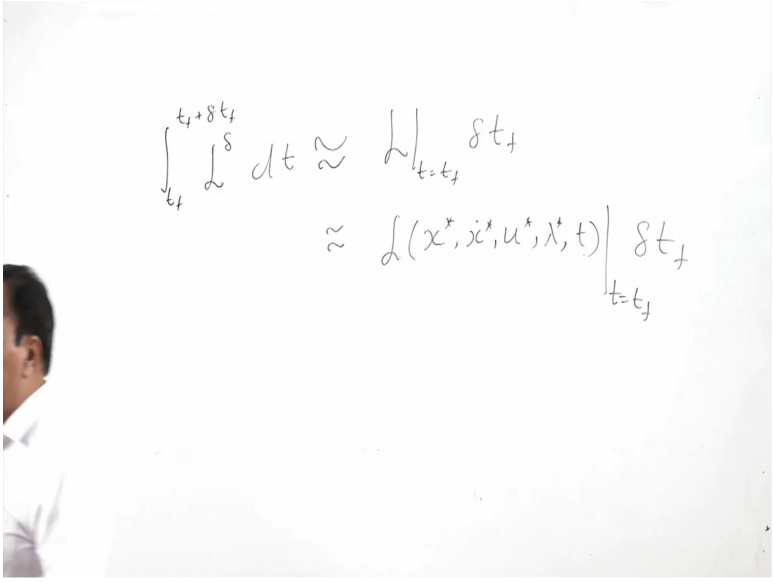

This means, this whole term I am approximating as L evaluated at t equal to t f point delta t f, which is equivalent to this means L which is a function of at x star, x dot star, u star, lambda star t at t equal to t f point delta t f. So, this is the meaning of this means this Lagrangian we are evaluating at optimal point and t equal to t f point.

So, in this we have treated this second term. So, second term in this can be written as Lagrangian at the t f point multiplied with the delta t f.

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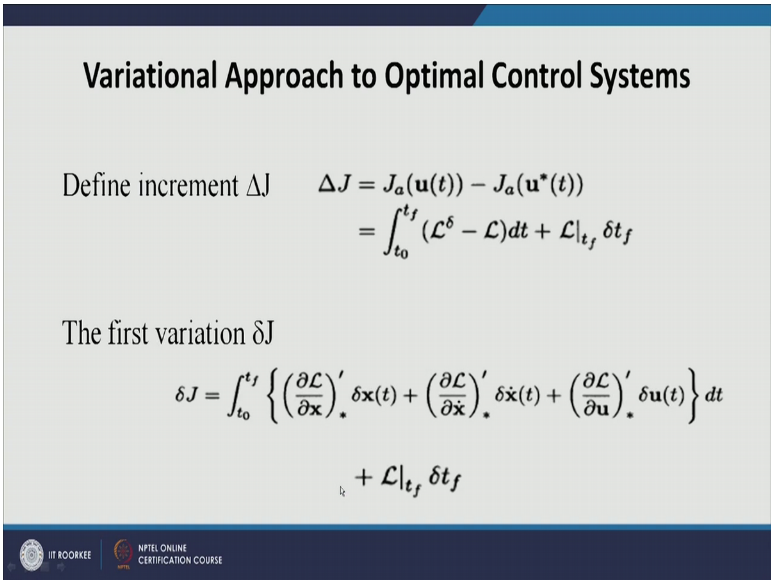

So, now, what we can have the increment delta J? This delta J can be written as value of the J at variation point minus value of the J at the optimal point. So, now, if you will see we are subtracting J u star from J u t. So, we are writing increment in J which is delta J has J u at variation point minus J u star. J a is my t 0 to t f L lambda plus the second term minus this. So, this two integral are from 0 t 0 to t f.

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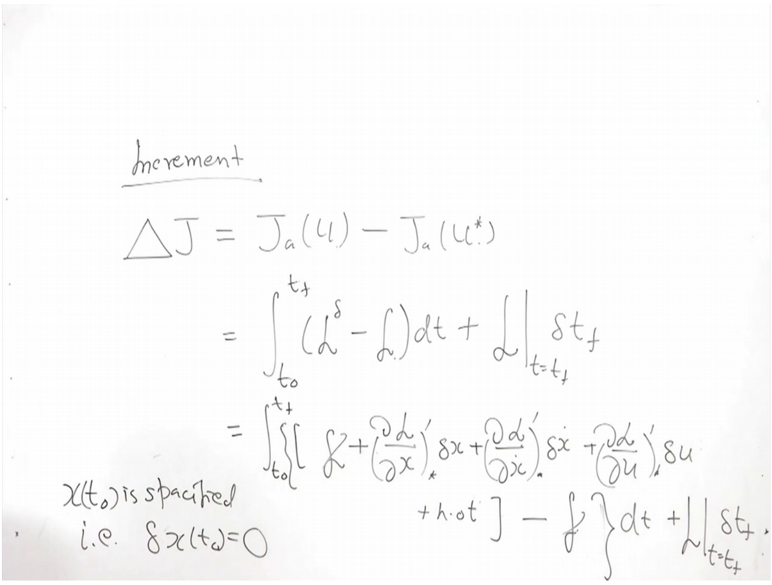

So, this is nothing but I can write t 0 to t f Lagrangian at variation point minus Lagrangian at optimal point dt plus my second term t f to t f plus delta t f, L lambda dt which we have a approximated as Lagrangian at t equal to t f point delta t f.

So, we got this as a variation; now again L lambda minus L this can be expanded using the Taylor series and then we can collect the first order term because this is the L at variation point this is the L at optimal point. So, naturally by variation will be. So, this L if I will explain this L lambda as L plus del L by del x. So, this L will be cancelled out, this means t 0 t f we are expanding this L delta as sorry L plus del L by del x to the optimal point delta x, del L by del x dot delta x dot plus del L by del u delta u naturally this all will be primes plus higher order term. So, this is the expansion of L lambda, this minus L sorry this we have taken as dt plus L t equal to t f point delta t here.

So, this L will cancelled out, we are collecting only the first order term and writing my first variation as delta J equal to t 0 to t f del L by d x, delta x del L by del x dot delta x dot del L by d u delta u into dt plus L t f delta t f. So, this first variation depend on delta x t, delta x dot t, delta u t. Now this delta x dot t I can write in terms of the delta x this means by integrating by parts the second term del L by del x dot delta x dot t dt this again I am integrating by parts.

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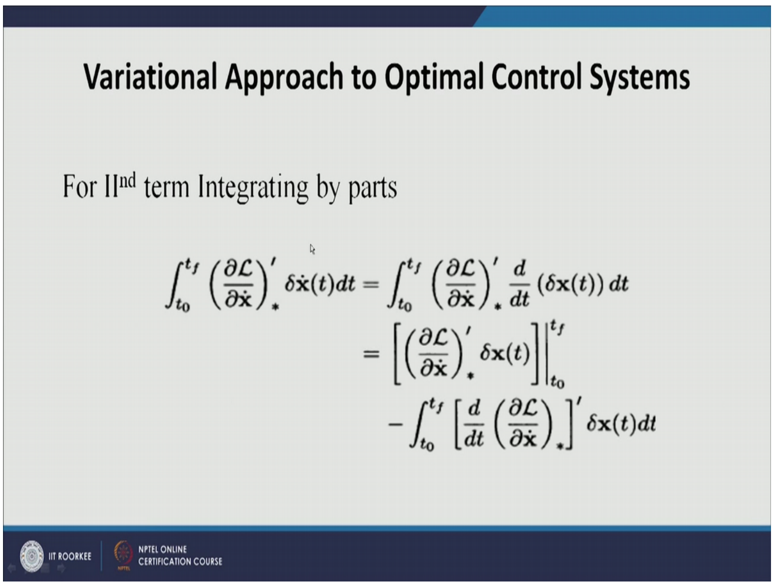

We already have done this before for two three cases. So, this can directly can be written as del L by del x dot, delta x this is my u this is my V, u V t 0 to t f means del L by del x dot delta x t, t 0 to t f minus d by dt of del L by del x dot delta x, and substituting this value in first variation term at this point.

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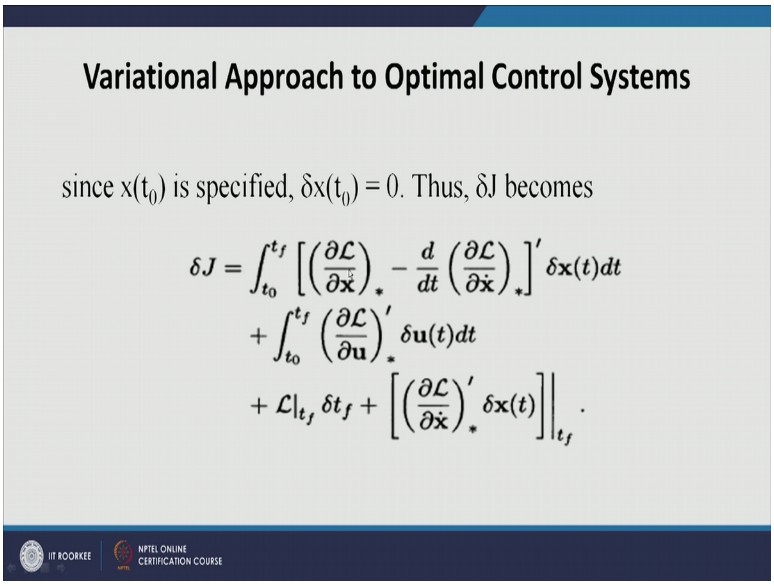

So, what actually we will get? Del L by del x minus d by dt del L by del x dot delta x t plus. So, the one integral is this second is t 0 to t f, del L by del u delta u t this we are getting from t f to t f plus delta t f, L of delta t f plus this condition we are getting from here once you are integrating by parts; because in this case what we have considered my initial condition is given. So, this means my t 0 and x of t 0 is specified, if x t 0 is specified this means x t 0 is specified this means delta x of t 0 is nothing but 0.

So, once we will expand my minus delta x t 0 this will become 0. So, I am only left with del L by del x dot delta x dot at t f point. So, L evaluated at the t f point, this term evaluated at the t f points. So, this will become give me my boundary condition. So now, if you will see this delta J, this will depend on delta x delta u and this boundary condition. So, again my variables are the x and u say normally in a control system if we have my input as a independent input I can give any input to my plant. So, my u will be the independent; if my u is the independent depending upon the u my all states are dependent on what the input I am giving into the system. So, from x and u, u is my independent variable while the x will be the dependent variable.

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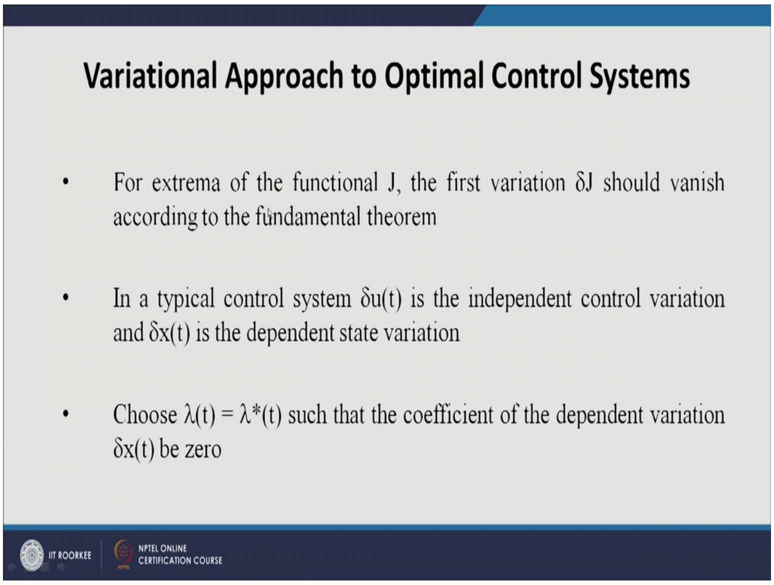

So, for extrema functional J, the first variation delta J should vanish according to fundamental theorem, u is independent delta x is dependent. So, what we can have? We can choose a lambda; I can choose my Lagrangian multiplier in such a way that the coefficient of the dependent variable will become 0. So, I can choose my lambda as lambda star so that the coefficient of the dependent variation delta x t will be 0.

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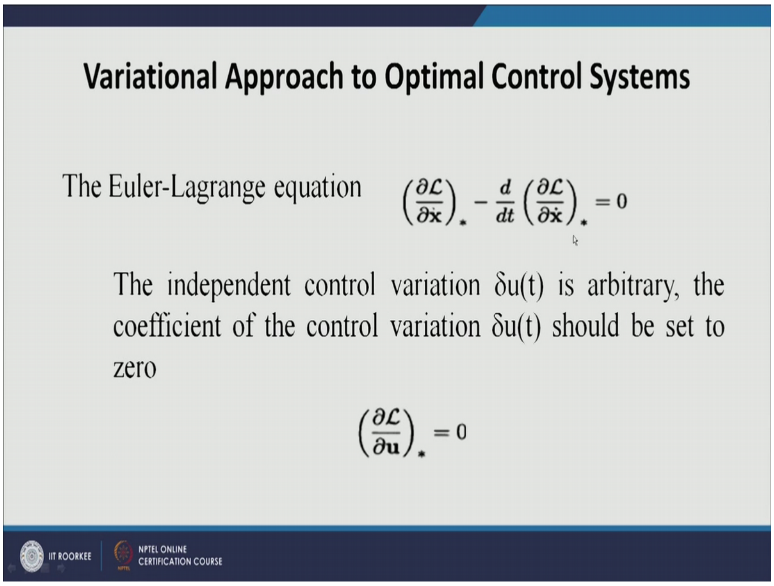

So, this means del L by del x minus d by dt, del L by del x dot equal to 0; and this is 0 because I have selected the lambda in such a way that it is going to make my this term to

0. If my this term is 0 I am only left with the next two terms del L by del u prime delta u dt, plus my condition.

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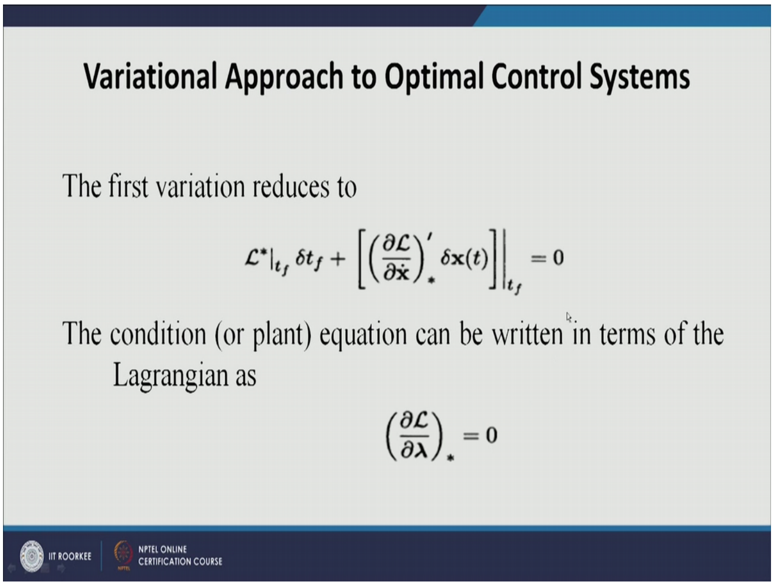

U is a independent variable delta u is arbitrary so, my coefficient related to this will be 0, and as a fundamental lemma I can directly right del L by del u will be 0. If del L by del u is 0. So, my condition here L t f delta t f plus del L by del x dot delta x dot this also will be 0, and because L is what? My integrant plus lambda prime g condition. So, del L by del lambda will coming out to be g which is nothing but 0. So, I can also write del L by del lambda equal to 0.

So, we got the four conditions the first del L by del x minus d by dt del L by del x dot equal to 0 this will give me the Euler equation, this will give me the control del L by del u equal to 0 means I am getting my control here, this is giving me the boundary condition and the third sorry the last one is del L by del lambda equal to 0.

So, this class I am stopping here next class we will continue from this point in which we will see how I can expand my final condition because this is depend on the delta t f, but here it is dependent the del x of t f; as t f is free so how will treat this case that we will see in the next class. So, this class I am stopping here.

Thank you very much.