Optimal Control
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Lecture - 10
Optimum of Functions with Conditions
(Lagrange Multiplier Method)

Welcome friends into the optimal control class. In the previous lecture we have discussed about the optimum of functions with condition and the approach which we have seen that was the direct approach. Say if a function is a variable of the, is a function of the 2 variable say x 1 and x 2. So, we will eliminate the one and then simply take the df by dx equal to 0 to find out the optimum point and d 2 f by dx square to get the sufficient condition if this is greater than 0 we get the minima if it is less than 0 we get the maxima.

Today we will continue the same topic optimum of functions with condition, but approach here we will take the Lagrangian approach.

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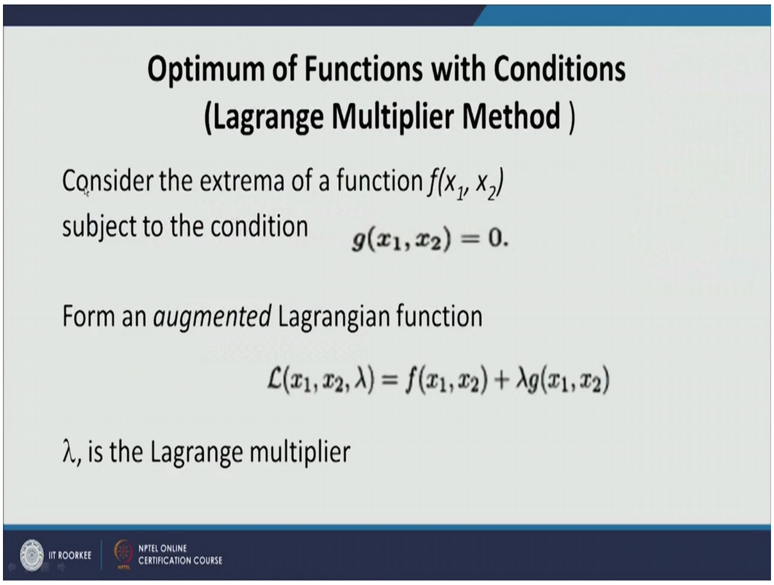

Let us see the problem our problem is we have a function, f which is the function of the 2 variables x 1 and x 2, subjected to a condition given as the g x 1, x 2 equal to 0 and objective is to get the optimum value of the function f with the given condition. In the Lagrangian approach we first define a augmented function called the Lagrangian function given as the L which is also the function of the 2 variable x 1, x 2 and the third variable we introduce that is lambda. And this is written as the f x 1, x 2 which is my given function plus lambda times whatever be the condition g x 1, x 2 lambda we called the Lagrangian multiplier. What is the role of the Lagrangian multiplier? That you will see in this lecture. Normally lambda is the user choice we have to select the lambda depending upon the condition arises. We will see in the next part.

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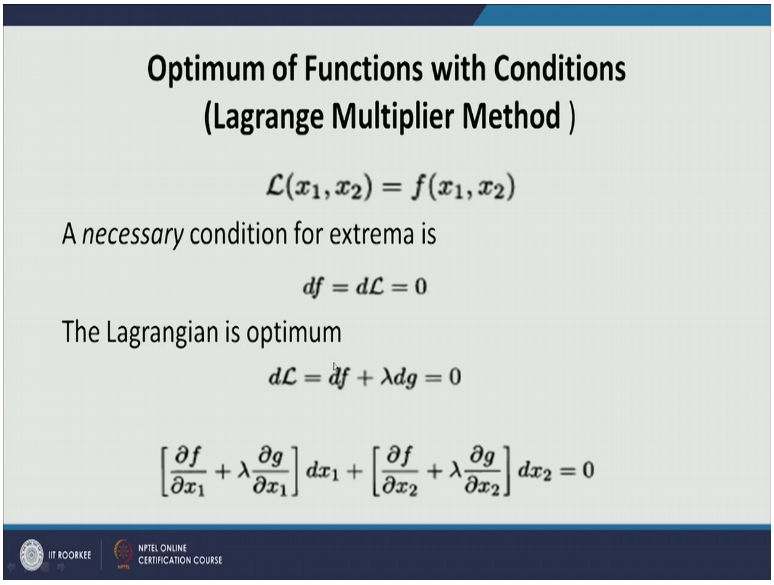

So, if you will see my Lagrangian is nothing but equal to f x 1, x 2 y because g x 1, x 2 is nothing but I have the 0 value. So, if I am writing L as f x 1, x 2. So, the minimization of the f was say the, to determine the optimal value of the f is same as to determine the optimal value of the L because my condition is 0. So, the necessary condition for an extrema, so that is df equal to dL equal to 0. So, I have to take the df the differential of my function must be equal to 0 because the optimal value of the f is same as the optimal value of the f, so we can say f equal to the differential of the L and that all must be equal to 0. Because f we have taken as sorry, L we have taken as the f plus lambda g. So, differential of the L we can write as the df plus lambda times dg.

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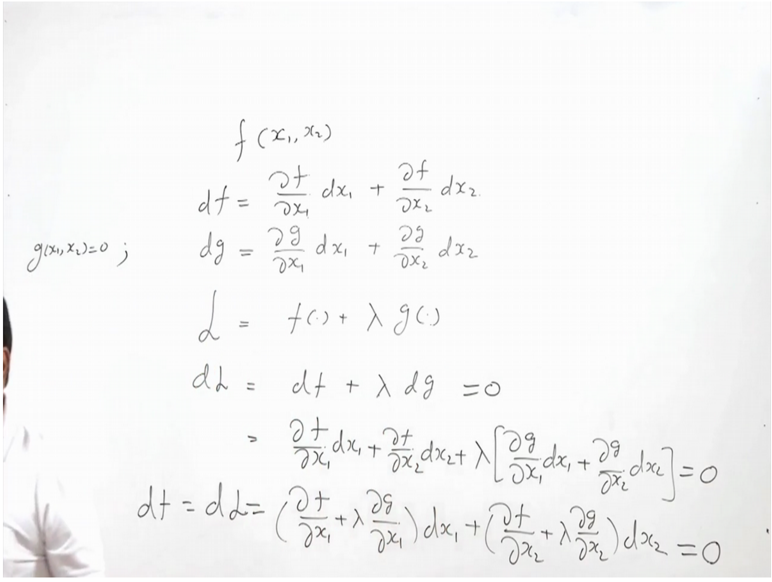

Now, if we will determine the value of the df and dg what actually we will get, that we can see because f is a function of x 1 and x 2. So, I can write my df is nothing but del f by del x 1 dx 1 plus del f by del x 2 dx 2. Similarly my dg what I can write and what actually I have g x 1, x 2 which is equal to 0. So, I can write dg as del g by del x 1 dx 1 plus del g by del x 2 dx 2 and what we are defining for L? That is nothing but f plus lambda times of g and dL we are writing as df plus lambda times d g. So, if I will place the value of df and dg what actually we will get del f by del x 1 dx 1 plus del f by del x 2 dx 2 plus lambda times del g by del x 1 dx 1 plus del g by del x 2 dx 2.

So, in this expression if I will collect the terms of the dx 1 and dx 2, so I can write this nothing but as del f by del x 1 plus lambda times del g by del x 1 dx 1 plus del f by del x 2 plus lambda times del g by del x 2, dx 2. So, this dL or we can say dL that I can represented as in terms of the dx 1 and dx 2 which we can see here. So, we are representing by this and this whole is equal to 0 because del is equal to 0, so this equal to 0 and this equal to 0. So, anyway we can say my del f by del x 1 plus lambda times del g by del x 1 multiplied with the dx 1, del f by del x 2 lambda times del g by del x 2 dx 2 equal to 0. So, this we are trying to get the necessary condition.

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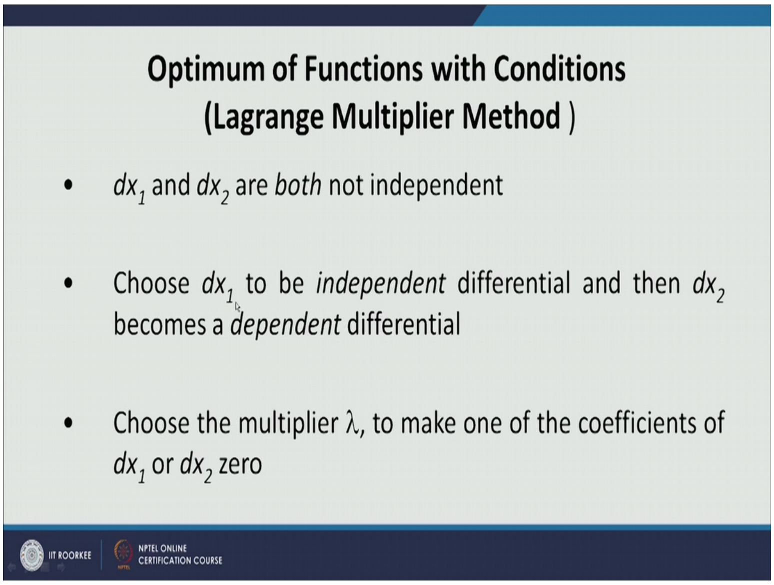

Now if we will see what actually x 1, x 2 x 1 and x 2 are not independent because this is the 2 variables related to each other. So, this means dx 1 and the dx 2 are also not independent, if we will choose one of the variable say x 1 is the independent variable then x 2 will be the dependent variable on x 1. So, I can choose dx 1 as my independent variable then dx 2 will be the dependent on x 2. Now here comes the role of the lambda what actually lambda we will take. So, this lambda we can choose the value of the lambda such that say in this case if I will choose lambda such that my coefficient of the dependent variable if this will become 0. So, that will be the, that will be my choice for the lambda.

At one take if having this if I will choose a lambda which will make the coefficient of the dx 2 as 0. So, my initial conditions simply will be which is dependent on the dx 1. So, that is why we say we have to choose lambda to make one of the coefficient of dx 1 or dx 2 to 0.

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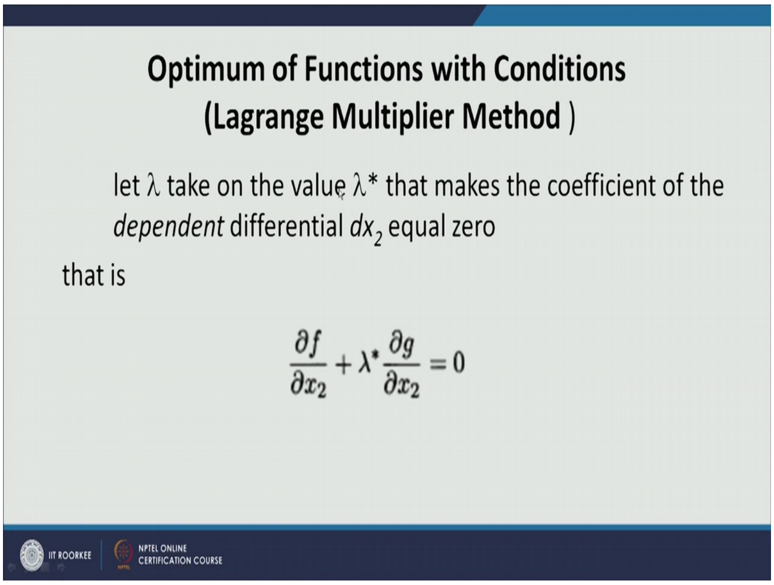

In this case we are selecting the lambda say the value lambda is star which will make the coefficient of the dx 2, 0. So, in that case what was my coefficient of the dx 2? This we are making 0, so this means del f by del x 2 plus lambda is taking the value of the lambda is star del g by del x 2 that will be equal to 0 if this term is 0.

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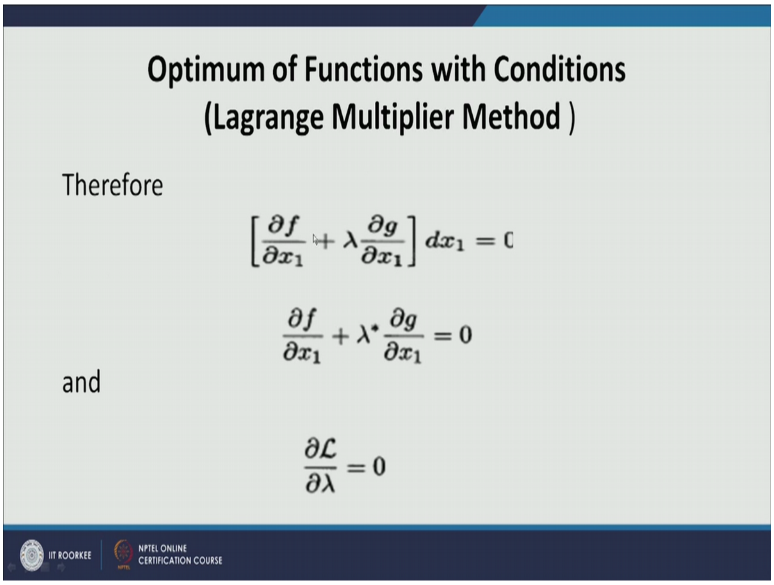

So, I will left only with the del f by del x 1 plus lambda times del g by del x 1 dx 1 which will be equal to 0.

Now, in this equation my dx 1 is a arbitrary function which can take any value. So, the coefficient of the dx 1 this will also be equal to 0. So, what was my condition? My first condition was the del f by del x 2 plus lambda times del g by del x 2 is 0, x is x 1 is the independent variable dx 1 is the arbitrary, so the coefficient of the dx 1 that will also be equal to 0.

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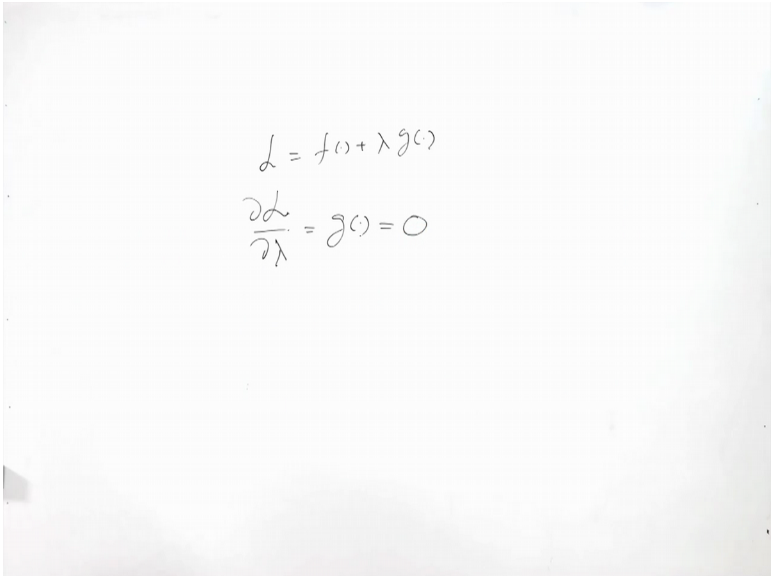

And if you will see what is my L, as my L is nothing but f plus lambda g. So, if I will take del L by del lambda this is nothing but my g which is 0. So, even I can write my last equation as del L by del lambda equal to 0.

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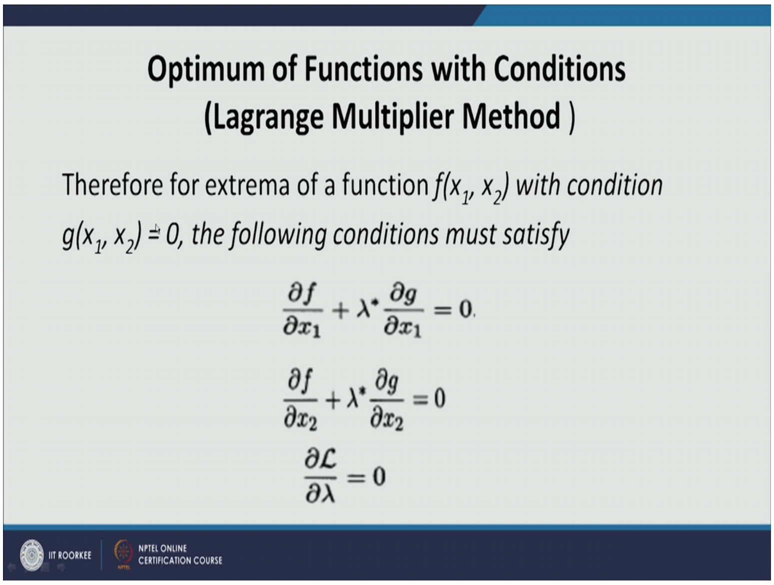

So, I have the three conditions now, if f is a function of x 1 and x 2. So, my to get the minimum my equation will be del f by del x 1 plus lambda ties del g by del x 1 equal to 0, del f by del x 2 plus lambda times del g by del x 2 equal to 0 and the third equation is del L by del lambda equal to 0.

So, to get the necessary condition we can apply this. So, these three equation will give me the x 1, x 2 and lambda these are the three variables which we have to determine. So, simply by writing this I will have the number of equation equal to number of variables and these can be solved to determine the value of the x 1, x 2 and lambda, and these will be my optimal values.

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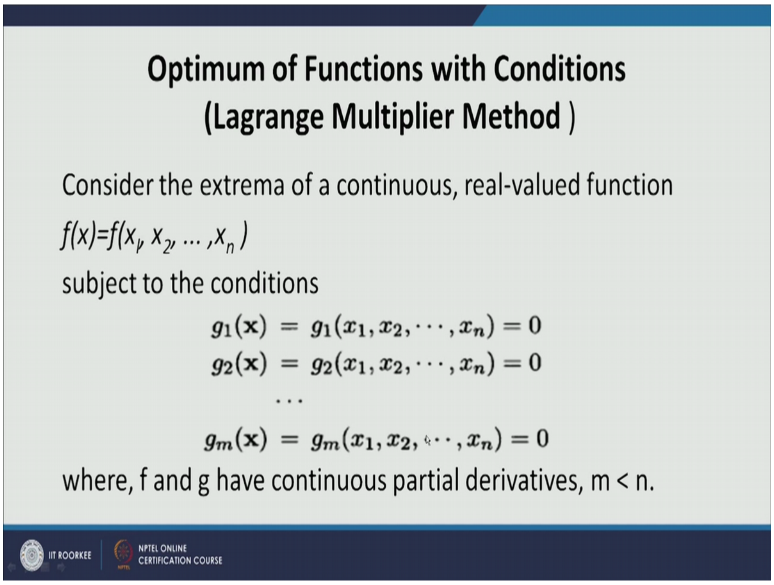

So, this was the case in which we are considering only the 2 variables x 1, x 2, but if we will generalize this case then suppose we consider a function which is a function of a vector x means f x is equal to f of x 1, x 2 and x n. So, it is a function of these n is states. So, this n optimal value we have to determine and this is subjected to the condition g 1 g 2 g m all are the function of x 1 to x n all equation. Where my g 1 is 0 g 2 x is 0 g m x is 0 and for a feasible solution the value of the m means the number of the condition must be less than the number of the variables which we have considered at the most they may be equal to each other, but m cannot be greater than n in that case my number of variables will be more than the number of equations and we are not able to find out the solution in that case. So, we put the condition that m should be less than n.

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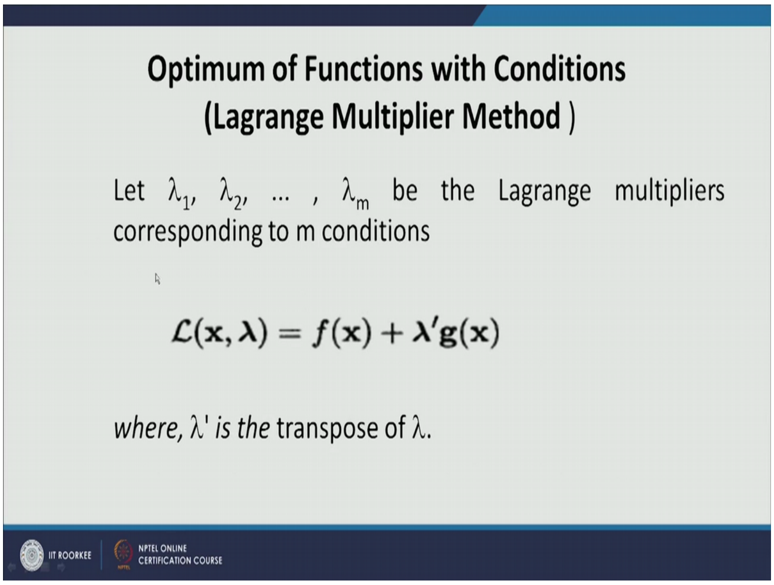

So, in this case also we can write our augmented function Lagrangian function, L as f x plus lambda prime g x where f is a function of x 1 to x m and lambda prime is a vector.

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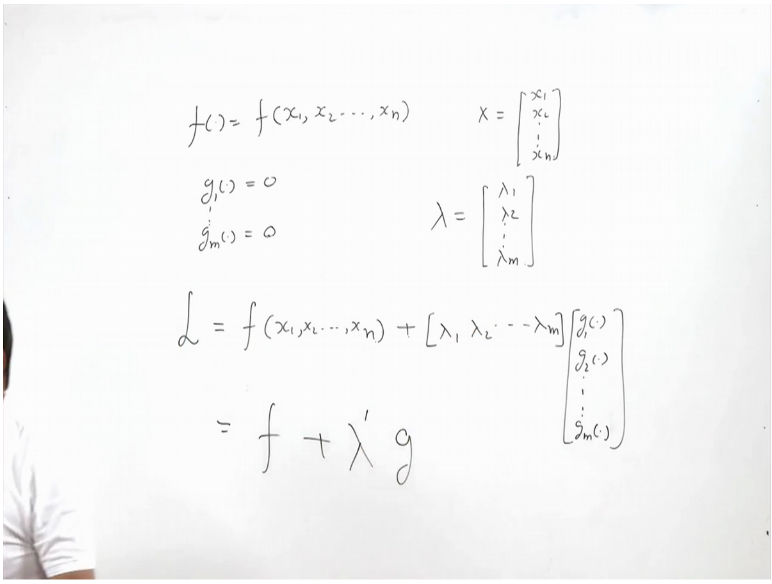

So, let us write this as my f is a function of x 1, x 2 and x n or we are saying my x is a vector of x 1, x 2 and x n. So, f will be the function of this. We have the conditions g 1 equal to 0 to g m equals to 0 I have the m number of the condition. So, the Lagrangian multiplier lambda which we have to select this will be nothing but a vector of lambda 1 lambda 2 each multiplier for each condition to lambda m. So, once we have to write my

Lagrangian this will be nothing but my f the function which is given x 1, x 2 sorry, this is equal to n, x n plus lambda prime; lambda prime we are writing is lambda 1 lambda 2 up to lambda m multiplied with the condition g 1 g 2 and up to g m. This means lambda 1 g 1 plus lambda 2 g 2 plus lambda m g m. So, that will be my augmented function L.

And this we are writing as f plus lambda prime g, where lambda prime is nothing but representing the vector lambda which we have selected here. So, prime represent here the transpose of lambda. So, transpose of the lambda will be lambda 1 to lambda m. So, we can write my augmented function as L equal to f plus lambda prime g. The optimal value x and lambda are the solution of my equations del L by del x del f by del x lambda prime del g by del x as we have seen in the first case del L by del lambda which is nothing but equal to g x that must be equal to 0.

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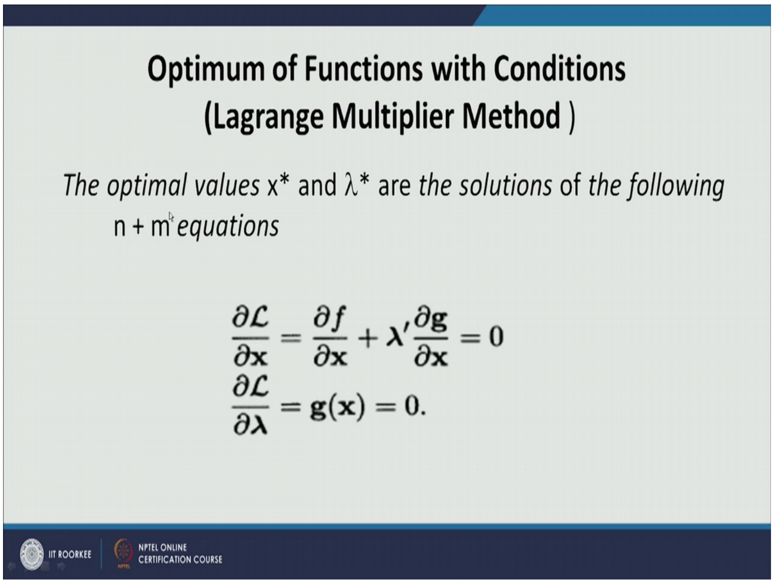

So, this the first del L by del x this is represent the n number of the equation because this is del L by del x 1, del L by del x 2, del L by del x n.

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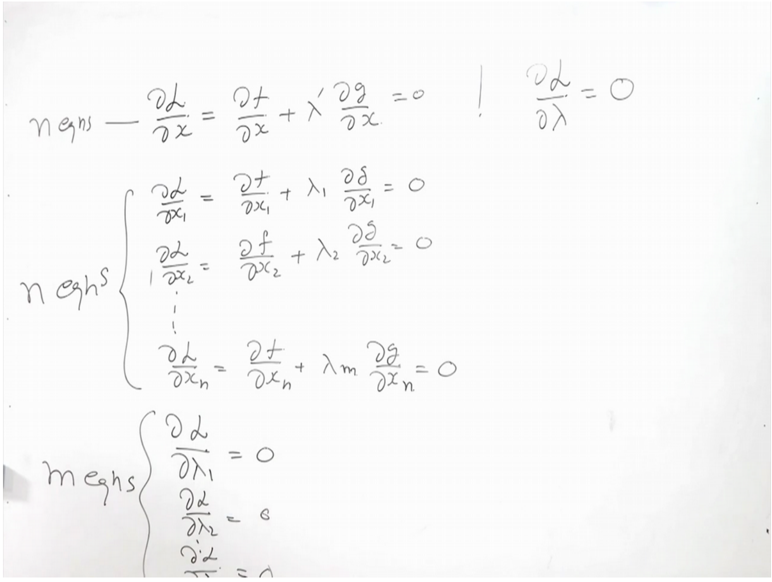

So, here if we are saying del L by del x equal to del f by del x plus lambda prime del g by del x and this we are saying equal to 0. So, this will contain n equation.

So, if we are taking del L by del x as del f by del x plus lambda prime del g by del x equal to 0. So, this means this is representing my n number of the equation which is equivalent to del L by del x 1, del f by del x 1 plus lambda 1, del g by del x 1 equal to 0 del L by del x 2, as del f by del x 2 plus lambda 2 del g by del x 2 equal to 0 and last equation del L by del x n as del f by del x n lambda m del g by del x m equal to 0. So, sorry, del g by del x n equal to 0 because g is a function of x 1 to x m f is a function of x 1 2 x n.

Number of Lagrangian multiplier is equal to number of condition which is given. So, these are the n number of the equation these are my n equations, n equations will come from my second condition which will say del L by del lambda equal to 0 this means del L by del lambda 1 equal to 0 del L by del lambda 2 equal to 0 and del L by del lambda m equal to 0 and this will give me m equations. So, I have total n plus m equations which is to be solved to find out the value of x 1 to x n and lambda 1 to lambda n.

So, what the values we are finding there, that all will be my optimal value. To understand clearly let us take one example which we have taken in the, for the direct method also.

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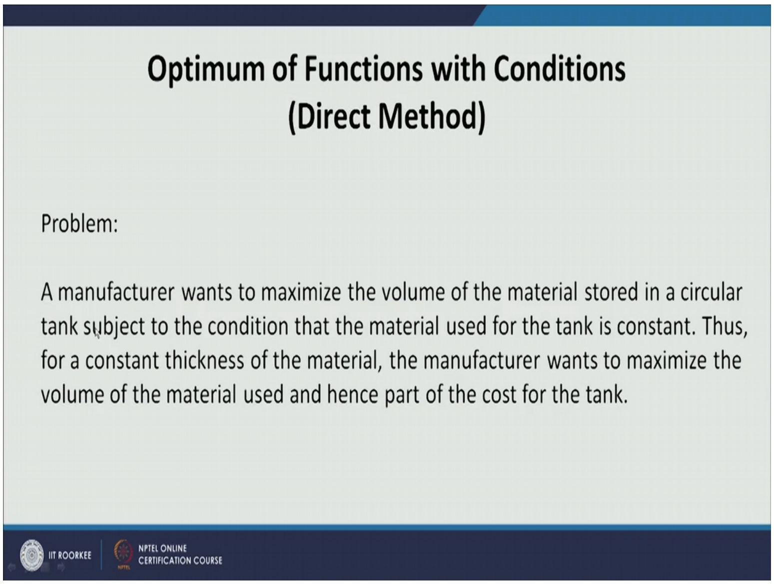

The same example we are taking here which is a manufacture want to maximize the volume of the material is stored in a circular tank which is subjected to the condition that material used in the tank is constant. Thus, for a constant thickness of the material, manufacture want to maximize the volume of the material used has he wants to simply reduce the cost, so in a minimum sorry; in the lower cost he can is store the maximum volume.

So, this is the same problem which we have discussed for the direct method approach. So, our objective here is to maximize the volume if we will consider d is the diameter, h is the height.

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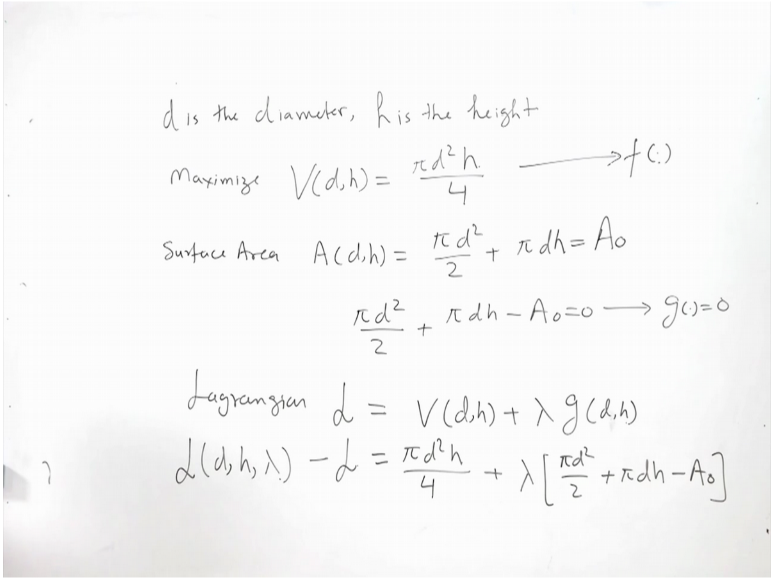

So, we have to maximize the volume which we are taking V which is a function of d and h and can be written as pi d square h by four let me change this and. So, I have to maximize this volume which will and surface area is a again a d h which is equal to pi d square by 2 plus pi d h. So, what is our problem? We have to maximize the volume if my diameter is d and height is h. So, this volume will be pi d square h by 4 my surface area is pi d square by 2 plus pi d h, what is my problem, my surface area is constant let say A naught.

So, I have to maximize my volume subjected to the condition for a constant surface area. So, what should be my d and h, so that my volume will be maximized? So, first we are solve this problem by the direct approach in which we are one of the variable, we are finding in terms of the A naught and substituting this value into the V taking d V by that particular variable equal to 0, so this give us the condition. In this case this is my f and condition I can write it as pi d square by 2 plus pi d h minus A naught equal to 0 and this is nothing but like my g equal to 0. So, as my first step I have to define my Lagrangian. So, my Lagrangian will be in this case f plus lambda g. So, I will have plus lambda g which again is a function of d and h, V I can write as pi d square h by 4 plus lambda times and g is pi d square by 2 plus pi d h minus A naught. So, this will be my Lagrangian.

So, this Lagrangian if I will say, this L is a function of d h and lambda. So, in my Lagrangian approach I have to take del L by del d equal to 0, del L by del h equal to 0, del L by del lambda equal to 0.

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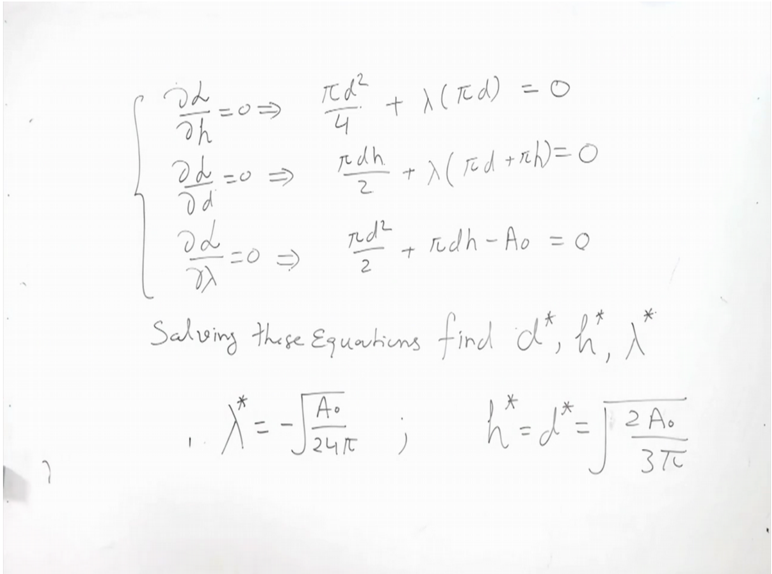

So, if L is this what I have? Del L by del h equal to 0 and this gives me with respect to a h I am differentiating del L by del h pi d square by 4 plus with respect to h I have to differentiate. So, pi lambda times pi d because this is independent of h this is independent of h, so I will left only the pi d that is equal to 0. My next condition is del L by del d equals to 0, I have to differentiate this with respect to d. So, this is giving me pi d h by 2, 2 d to 2 this will give me pi dg by 2 plus lambda times pi d and this will give me the pi h and this equal to 0 and my third condition is del L by del lambda equal to 0 and this will give me nothing but whatever we my condition which is pi d square by 2 plus pi d h minus A naught equal to 0.

So, these three equations n is 2 here I have the 2 variable, m I have the one condition. So, as the total three equations; I have the three variables d h and lambda, these three I can find out by solving this equation. So, solving these equations find t star h star and lambda star. So, the solution of the equation gives me lambda star is A naught by 24 pi and as we have find in the previous case my h height is equal to diameter and this is equal to 2 A naught by 3 pi. So, in this way we can find out the optimum of a function which is subjected to a given condition.

So, simply we have to write first the augmented Lagrangian function which is nothing but my function which I have to minimize f plus lambda times what is my condition g then I have to differentiate L with respect to all my variables. Like in this case d and h versus the x 1, x 2 x n del L by del x 1 equal to 0, del L by del x 2 equal to 0 and similarly for all the lambdas.

So, today's lecture we stop here and in the next class we will start our discussion on the optimum of the functional. So, till now we have discussed only the functions and in the next class we will start our optimum of the functional with the given conditions.

Thank you very much.