Stochastic Modeling and the Theory of Queues
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Lecture-80
Semi Markov Processes-Part 1

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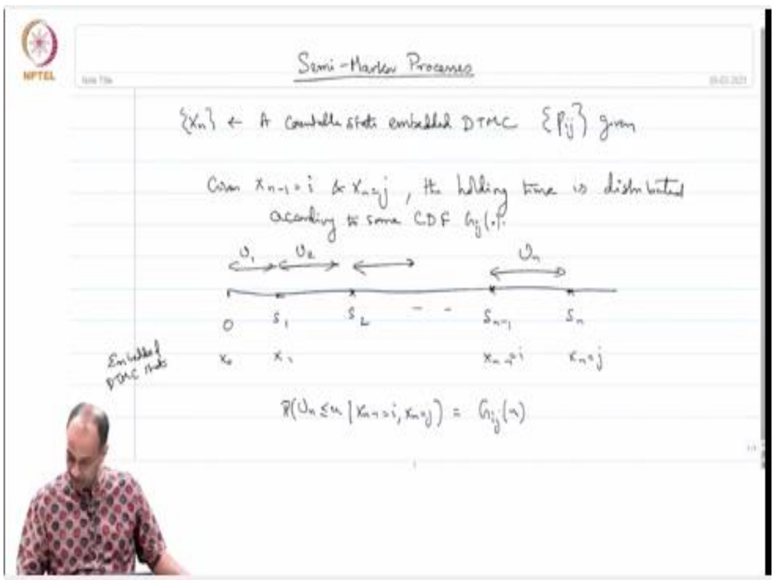

Welcome back this is the last module of the course. So far we have discussed CTMC's continuous time Markov chains. Today we will discuss semi-Markov processes which are like a generalization of CTMCs as well as in fact they are also generalization of renewal processes. So, a semi-Markov process is characterized by 2 things, so you are given DTMC, let us say accountable state DTMC, embedded DTMC.

So, X n is a countable state embedded DTMC just like for CTMCs. And then you have holding times which are not necessarily exponential. For a Markov process in each state you have a exponential holding time. So, if you are in state j you hold for exponential nu j. In a semi-Markov process you can hold for any general distribution. And the way it works is that, so you are let us say given X n - 1 = j or maybe it is X n - 1 = i and X n = j. So, this happens with probability P ij.

So, this transition probability P ij is given, the holding time is distributed according to some CDF G ij. So, let me should draw a picture, so you have let us say that is time 0 and so you have this is S 1, where the first transition occurs is S 2 dot, dot, dot S n – 1, S n and these we will denote by U 1 just like in CTMC's and so on. So, this will be U n and of course the state transitions here will be, so the states of the DTMC, embedded DTMC state X 0, X 1, X n – 1, X n.

So, S n's are the epoch, S n is the epoch where the nth transition takes place and X n is the state of the embedded DTMC at the nth transition. So, you can say that, so let us say if X n is in some state i, X n - 1 is in state i and X n = state j. Then probability of U n less than or equal to u given X n - 1 = i, X n = j is just some G ij of u, this is some CDF. So, notice that we are allowing the holding time to not just depend on which state you are in, it could also depend on which state you are going to.

Of course the case where the holding time does not depend on j is a special case of this. And in fact in the CTMC case the holding time depended only on the state that you are in, not which state you are going to. But in the semi-Markov process we can even allow that, I mean there is no problem in allowing this. Now the dependent structure, I will tell you little bit about these dependences, what we are going to say is that this U n random variable. So, let me write this down, the dependent structure.

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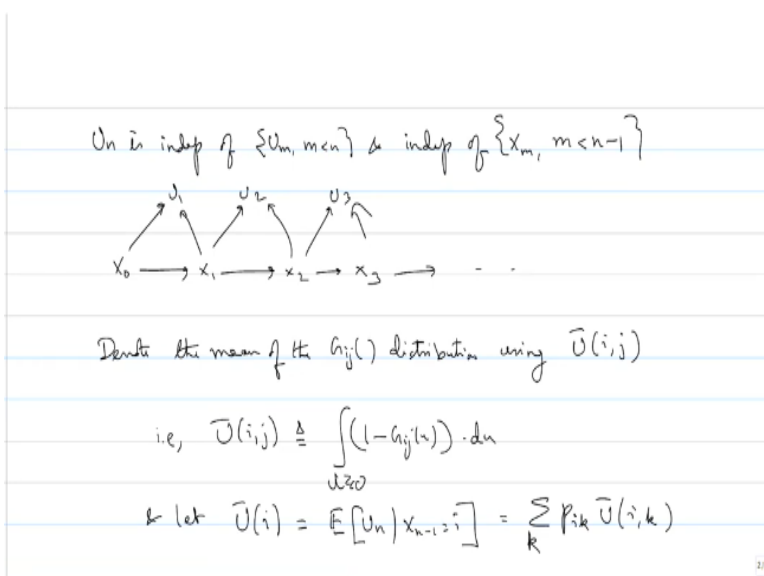

So, this is for u greater than or equal to 0 and U n is independent of U m for m less than n and independent of X m for m less than n - 1. So, the dependent structure is like this, so you have X 0, then X 1. And this 2 X 0 is the first state of the embedded DTMC, the X 1 is the second state, these 2 together can determine U 1. And X 2, so X 0, X 1, X 2 etcetera evolves according to a DTMC, that we already know.

So, condition on X 1, X 0 and X 2 are independent, that we already know. Now the dependence of U's are such that, now U 2 dependence is only between X 1 and X 2, so U 2 is independent of U 1. And U 2 is also independent of X 0 and generally U n is independent of X n - 2 and so on. So, this is the dependent structure dot, dot, dot. So, we will denote the mean of the G ij distribution using U bar ij.

So, in particular ie U bar ij is just integral U greater than or equal to 0 1 - G ij of u, so I am just integrating the complementary CDF. I can integrate the complementary CDF to get the mean that is because this is a non-negative random variable. So, this is just a notation, U bar ij is the mean holding time in state i if you are going to state j next. And let U bar j or U bar let us say U bar i as expected U n given X n - 1 = i, this is the holding time in state i.

Now unconditioning on what the next state is? This will simply be sum over p i, k U bar i, k sum over all these states. So, I am just taking, so U bar i, k is the expected holding time in i given that you are going to k. Now I am averaging over all the p ik’s this is a just total expectation. So, this is the average holding time in state i, unconditional of where you are going. So, these 2 are useful notations.

Now everything that we really need to know about these semi-Markov processes in terms of time average and all that follow similarly to the CTMC case. So, if you want to look at the average fraction of time spent in some state j. In the CTMC case we got if pi k was the underlying distribution of the embedded Markov chain, we could get P j as pi j over nu j divided by sum over pi k over nu k.

Assuming that the denominator is finite, actually even if the denominator is infinite it is still correct; except you have P ij is 0 in that case. Similar something very similar holds here the arguments are almost identical. The point is that if you look at M ij of t as you did in the CTMC case.

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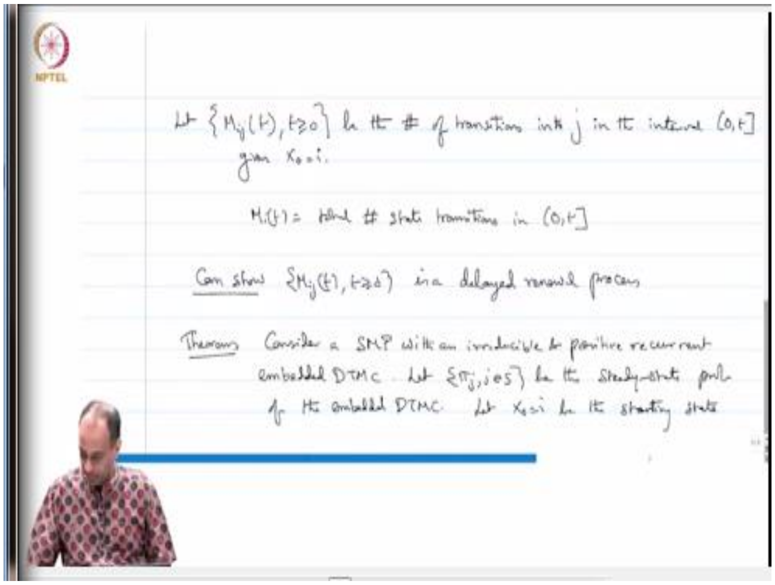

Say let if you look at this counting process be the number of transitions into j given X 0 = i in the interval 0, t given X 0 = i. And m i of t = total number of state transitions by the embedded DTMC in 0, t. These are exactly the same notations we use just for the CTMC case. What we can show is that, M ij of t is a delayed renewal process. So, you are going from i to j first and then you are going from j to j again and again. So, successive entries into state j will be iid independent and identically distributed.

That is because of the strong Markov property of the embedded Markov chain and the fact that the U random variables have a certain independence property as well. Using these 2 facts I mean this requires a proof, I mean I am not getting into the proof of this in great detail but you can show this. First of all you have to show that these successive transitions happen in finite time with probability 1 and all that requires a proof.

But I mean they are essentially similar to what we did for CTMCs. So, you can show that M ij of t is a delayed renewal process using the definition of the semi-Markov process. So, you can use the same machinery to show a theorem like this which is analogous to what we did for time average occupancy of states in the CTMC. Consider a semi-Markov process with an irreducible and positive recurrent embedded DTMC.

And let pi j be the steady state probabilities of the embedded DTMC. So, you are taking a nice embedded Markov chain which is irreducible and positive recurrent and it has pi j as it is steady state probabilities. And you are taking let us say some let X 0 be the starting state X 0 = i be the starting state.

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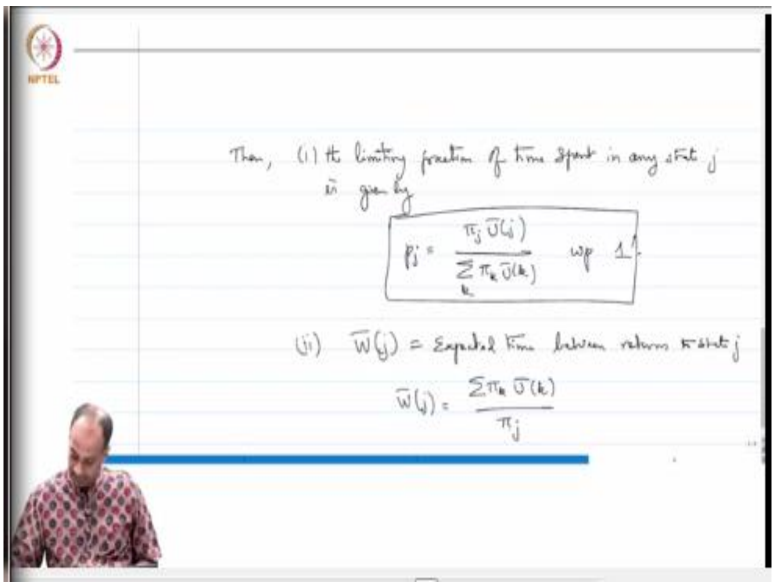

Then number 1, the limiting fraction of time spent in any state j is given by p j = pi j U bar j over sum over k pi k U bar k with probability 1. This is a fraction of time spent in state j this is almost surely equal to pi j U bar j over sum over pi k U bar k. This is a valid probability distribution if the denominator is finite. If the denominator is infinite you will get 0, there is no meaningful notion of fraction of time spent in any state.

So, if you think about this, this U bar j is playing the generalized role of 1 over nu j in the CTMC case. So, the CTMC case you got pi j over nu j divided by sum over pi k over nu k. So, 1 over nu j is simply U bar j. Remember that U bar j was defined here, it is the average holding time in state i irrespective of where you are going. So, U bar ij does not directly matter, although we allow this U n to depend on i as well as the next state j, U bar ij does not directly matter.

What really matters for this average fraction of time is this U bar j U bar i. So, this is all from renewal, this is along the same lines as the proof for CTMC's. If you think about it in the CTMC case we had a reward of 1 for whenever you are in state i. So, we consider the delayed annual process M ij of t and whenever you are in state, whatever state j you are interested in you give a reward of 1.

And it turned out that the average reward in that state was 1 over nu j. Now that 1 over nu j will simply be U bar j, that is all and everything else follows identically. We can also derive the expected renewal interval of successive returns to state j which is W bar j, this is the expected time between returns to state j which is equal to sum over pi k U bar k over pi j. So, remember that pi j W bar j was constant in the CTMC case, it turned out to be some constant and which we could find out later and something similar holds even in the semi Markov case.

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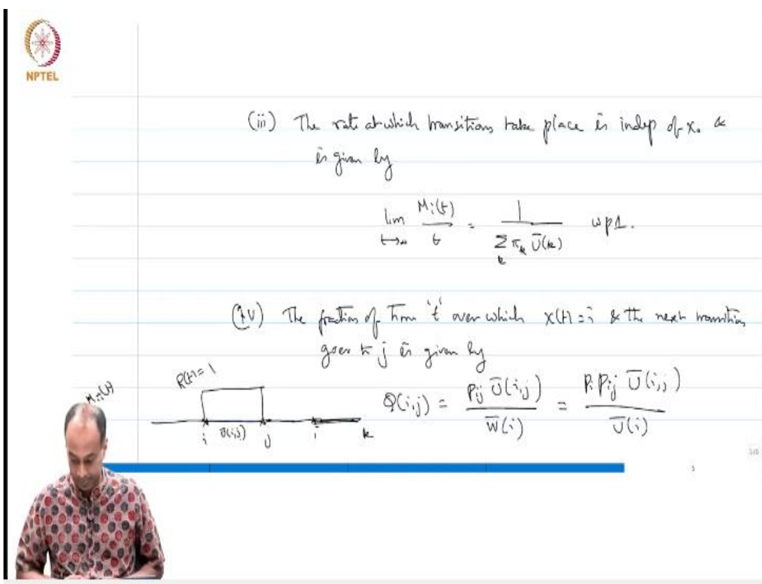

And finally we can write out the rate of transitions. Transitions takes place is independent of X 0, it does not matter which state you start in and is given by what is the average rate of transition? It is simply limit t tending to infinity M i of t over t which is just 1 over sum over k pi k U bar k. So, remember in the CTMC case you got pi k 1 over sum over k pi k over nu k, here you are getting instead of 1 over nu k U bar k.

This is also almost surely or with probability 1 whichever you like. So, these are analogous it follows almost identically to the CTMC proof for the corresponding quantities. So, I will not bother getting into the proofs again. So, we have the average fraction of time spent in state j, average rate of transitions, average time between transitions to state j and all that worked out. There is one more thing that is of interest in semi Markov process which was not very important in the CTMC case.

It is the average fraction of time that maybe I should also state this as a part of the theorem which also comes from renewal reward. We should, the fraction of time let me just state it first time t over which X t = i and the next transition goes to j is given by let us call that Q ij. It is given by P ij of the DTMC U bar ij over W bar of i which if you rearrange becomes we know the expression for W bar i from that.

If you just put that back in, so you want to write we do not know what W bar is, it is not specified as such in the process. So, we need to use this expression to get it in terms of the U bars. So, you will get P i P ij U bar ij over U bar i. So, the fraction of time t, so if you look at let us say you are in state i here, you could go to some other state. So, you know the fraction of time you are in state i is already known which is P i.

But you want to look at the fraction of time for which you are in i and the next transition goes to j. So, you give a reward of 1 let us say let us call this R ij of t or some sets. Let us call this you consider the renewal process M j of t, so you look at the renewal process let us say M ii of t, let us say this is a renewal process. And you give a reward of R t = 1 whenever you are in state i and the next transition is to state j.

But if you are in state i and the next transition is to state k you do not give any reward. You get 0 rewards, you would consider this reward process you can easily calculate the fraction of time that the fraction of time you spend in state i transitioning into state j, that is comes out like this. So, this is clearly I mean this is the w bar i in the denominator is X bar for this renewal process. And P ij is the probability with which that U bar ij is the average holding time, whenever you are in state i transitioning to j.

But you also have a P ij factor because you could go into any one of the other states as well. P ij is the probability with which you go from i to j in that case you collect a reward of U bar ij, that is the duration in which you stay, this is typically U bar ij duration, the expected duration is U bar ij, from this you can calculate this quantity. So, these things you can calculate using a renewal processes.