Stochastic Modeling and the Theory of Queues
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Module - 1
Lecture - 8
Poisson Process - Increment Properties

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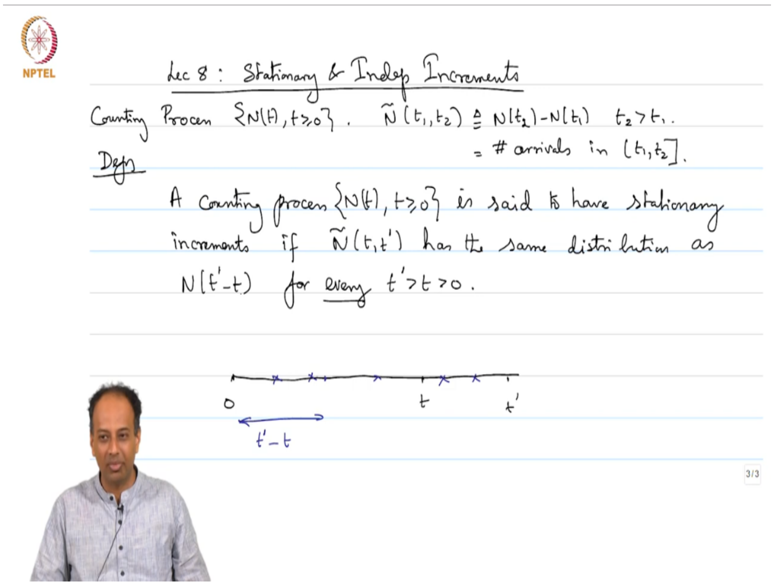

Stationary and Independent Increments. Definition: So, as a matter of notation, let me just develop one notation right now. So, $\{N(t),\ t\ \geq 0\}$we know. So, $\{N(t),\ t\ \geq 0\}$ is a counting process. Say, counting process $\{N(t),\ t\ \geq 0\}$is given. Now, I want to call $\tilde{N}(t_1, t_2)\ = N(t_2)\ -\ N(t_1),\ t_2\ > t_1$. So, this is just the number of arrivals in $(t_1,\ t_2]$, just notation. $\tilde{N}(t_1, t_2)$ is the number of arrivals that occurred in the interval $(t_1,\ t_2]$ (t_1 not included, t_2 included, just a convention).

Now, definition: A counting process $\{N(t),\ t\ \geq 0\}$is said to have stationary increments if $\tilde{N}(t\ , t'\ )$has the same distribution as $N(t'\ -\ t)$, for every $t'\ >\ t\ >\ 0$, be true for **every**, this **every** is important. So, this is a property called stationary increment property or for short, SIT, stationary increment property. So, we say that, the counting process $\{N(t),\ t\ \geq 0\}$has stationary increments if the number of arrivals in any time interval, let us say $(t,\ t']$, the number of arrivals in that interval has the same distribution as the number of arrivals in $(0,\ t'\ -\ t]$.

In other words, the distribution of the number of arrivals in any interval, depends only on how long the interval is. So, if you look at; so, if this is your time axis; this is 0. Let us say, that is t; that is t'. And you have some arrivals. You just look at the number of arrivals in this interval $(t, t']$, and you look at another interval of the same width, but starting at the origin. Say, this guy is $t' - t$; okay? So, you take these two random variables.

So, the number of arrivals that occurred in $(t, t']$, number of arrivals that occur in $(0, t' - t]$. So, you are considering two intervals of the same width. One is located at the origin, one is located anywhere else. If the distribution of the number of arrivals is the same for any t' in t, then you say that the process has stationary increments. So, what I really mean is, you know; yeah, thanks for pointing out; I should be saying; let me just erase this.

I should really be saying $\tilde{N}(t, t')$. Or I should say $\tilde{N}(0, t' - t)$, which is the same as $N(t' - t)$. So, it does not matter where the interval is located, the number of arrivals that occur in an interval only depends on the width of the interval and not where the interval is located. This interval could be at the origin or anywhere else. As long as the width is fixed, the distribution of the number of arrivals in that interval is fixed.

I mean, I am not saying that this is always true, I am saying that, if this is true, then such a process is said to have stationary increment property. And we will define independent increments also.

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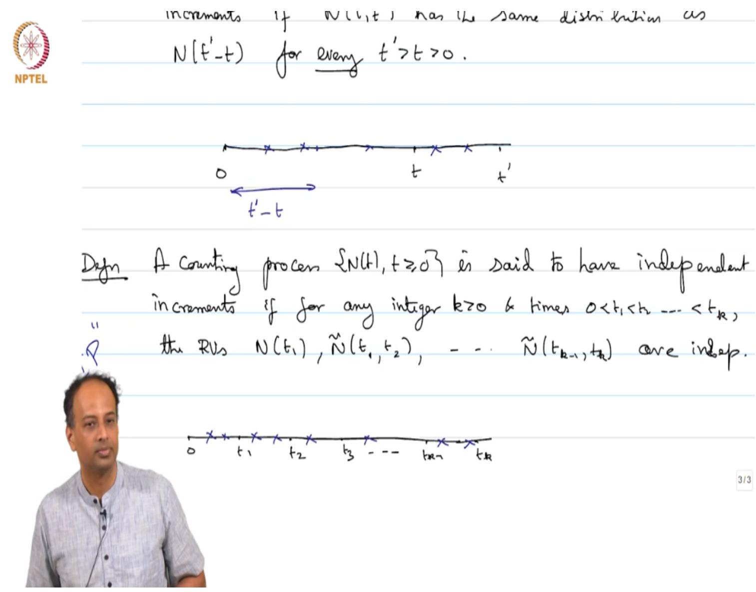

Definition: A counting process $\{N(t),\ t\ \geq 0\}$ is said to have independent increments if for any integer $k\ >\ 0$ and times $0\ <\ t_1\ <\ t_2\ <\ldots<\ t_k$, the random variables $N(t_1)$, $\tilde{N}(t_2,\ t_1)$; or I should say $\tilde{N}(t_1,\ t_2)$ I think, I am sorry, the smaller time always first; go on till $\tilde{N}(t_{k-1},\ t_k)$ are independent. So, you are looking at; good to just go back to this picture again. So, that is 0, and you are looking at times $t_1,\ t_2,\ t_3,\ldots t_{k-1},\ t_k$.

So, you can choose any k you want, and any t_k's, okay? You choose them whichever way you want, but they are fixed. And then you look at the; then, of course the process is running, the counting process is running. Then you look at $N(t_1)$, $\tilde{N}(t_1, t_2)$. Basically, you are looking at the number of arrivals that occur in each of these intervals. So, this will be some random variable, this will be some random variable and so on.

So, you get how many random variables now? k or? k, right? You get k random variables, this $N(t_1)$, $\tilde{N}(t_1, t_2)$ and so on. These k random variables should be independent, no matter how you choose the t's. We can choose any number; k can be anything and $t_1,\ t_2,\ t_3,\ldots t_k$ can be anything. For every such k and $t_1,\ t_2,\ \ldots,\ t_k$, if this k tuple of random variables are independent, then the process is set to have independent increment property, IIP for short.

k random variables being independent, you know what it means, right? The joint CDF will factorise into the product of the marginals. This should be true for every k and every

$t_1, t_2, \ldots, t_k$. If it is not true, even for one such tuple, then, once it is selection of the times, then the process does not have IIP. So, we have just defined two properties. One is the stationary increment property, one is the independent increment property.

It is of course not at all the case that a counting process should have either of these. It could have 1, not the other or neither. Now, what we are going to say next is that the Poisson process has both properties. The Poisson process has stationary increments and independent increments.

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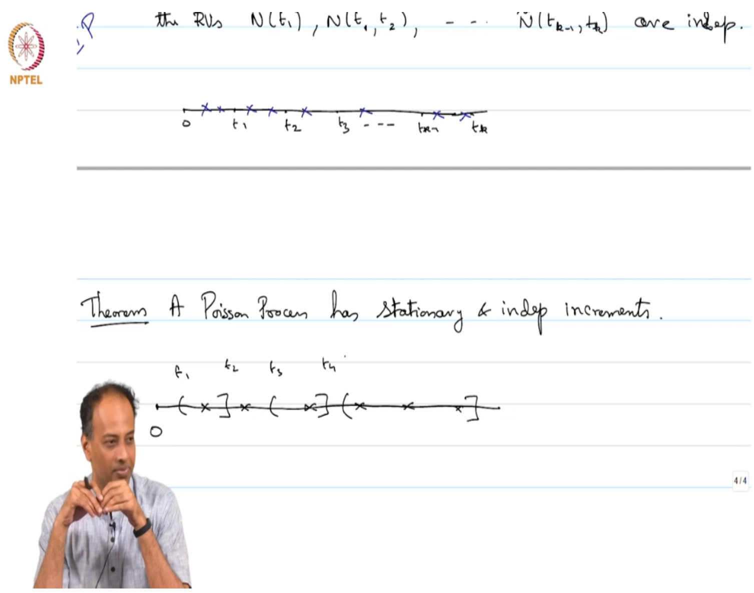

A Poisson process has stationary and independent increments. So, this theorem is actually a consequence of the memoryless property that we proved in the previous; so, you know, you remember this theorem, right?

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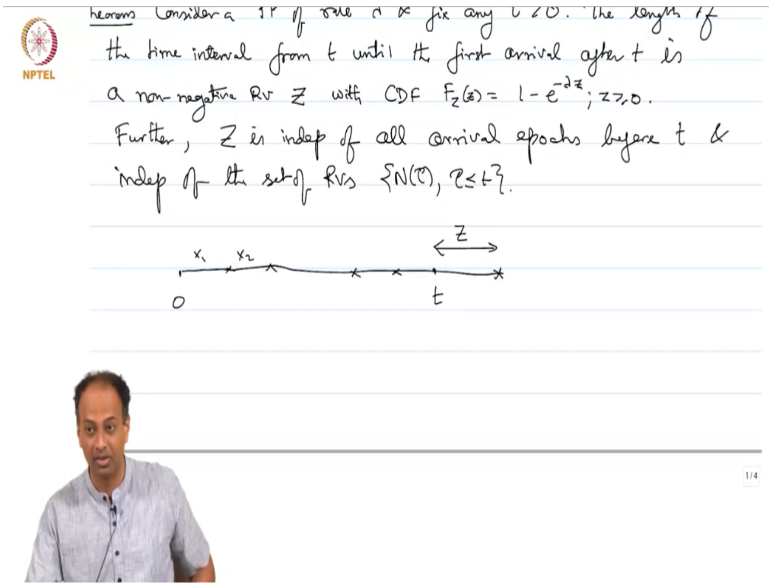

So, it is a consequence of; both IIP and SIP are consequences of this theorem.

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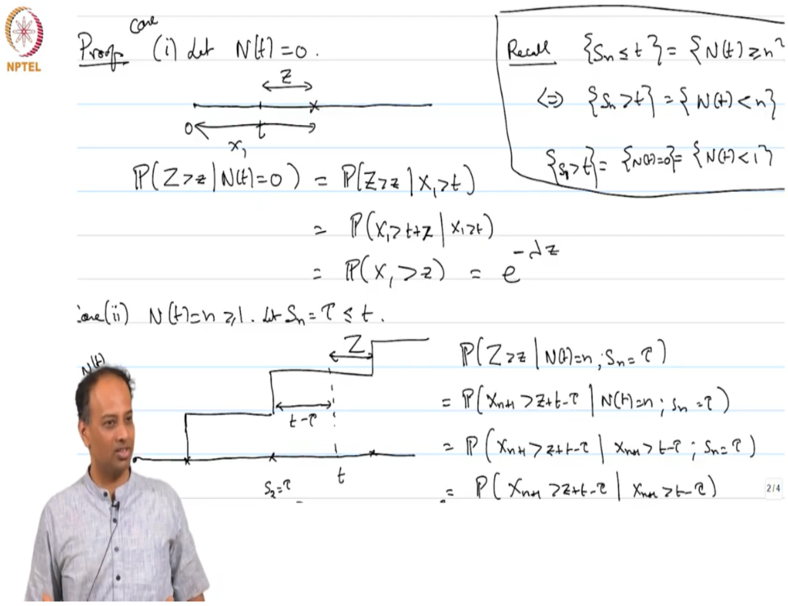

So, in particular, if you want to look at the stationary increment property, you are starting at some time t and looking at, let us say, the number of arrivals that occur from $(t,\ t']$. But we just proved in the theorem that the process restarts at t; it forgets everything that happened in the past. It is as though the Poisson process is starting at t. So, that t sort of behaves like 0.

So, the number of arrivals that occur in the time $(t,\ t']$, will be the same as the number of arrivals, statistically distributed; distribution wise, the same as the number of arrivals that occur in $(0,\ t' - t]$.So, this can be formalised, using the theorem that we proved earlier.

Clear? So, stationary increments are a direct consequence of this theorem. Independent increments are also a consequence of this theorem. It requires a little more argument.

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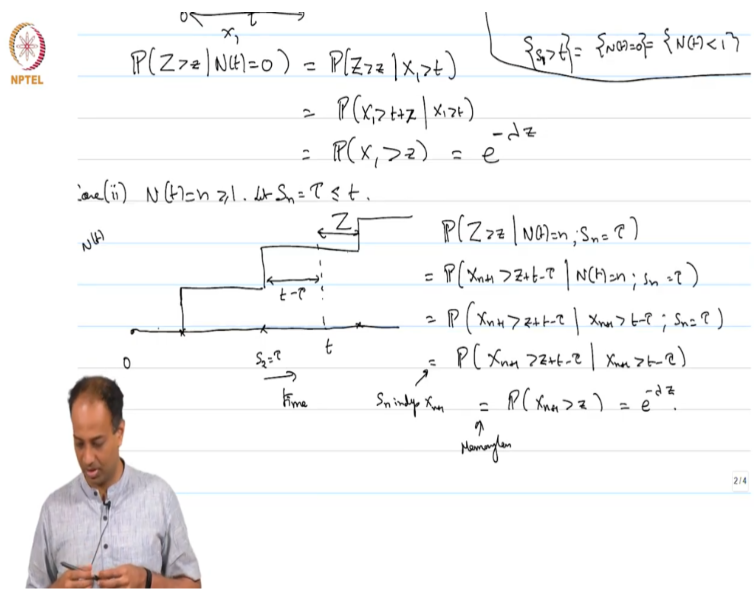

So, in fact, if you look at this argument, I mentioned that the; so, if you are here at t and if you are looking at the number of arrivals in $(t,\ t']$, we want to prove that, that is independent of the, any; if you mark $t_1,\ t_2$,..., et cetera here. Basically, you can argue that; you have shown that this Z, the time to the next subsequent arrival is independent of all the $N(t)$ here.

So, you have proven in this, essentially in this argument that Z is independent of $N(t_1)$, $N(t_2)$,$\tilde{N}(t_1, t_2)$, whatever, you marked those times here. You just have to prove that the number of arrivals in $(t,\ t']$, any t' that you mark out here, is independent of the arrivals that have come so far. That will come from the property of Z being independent of everything in the past, and the subsequent arrivals are anyway independent of everything that happened in the past.

So, you can put these two together and make this formal. Actually, your book, Gallager's book has this argument written out, but the; this is all there is, there is nothing more to it than just, in this theorem. It is just a consequence of what we proved. So, that we; so, we get this very important result. A Poisson process has stationary and independent increments. So, what that means is that, if I take a Poisson process; so, you have some Poisson process running.

Because of the stationary increment property, if you take any interval like; so, let us say this is t_1, this is t_2. So, the number of arrivals in an interval like this will only depend on the width of the interval; it does not matter where the interval is located. It will depend only on $t_2 - t_1$, it will not depend on the actual values of t_1 and t_2. So, this interval might have been here or it does not matter where it is; that is what stationary increment is saying.

And the independent increments is saying that if you consider many intervals like that, which are non-overlapping; basically, there is, if the intervals are disjoint, then the random variables corresponding to the number of arrivals in those intervals will be independent. So, all I am saying is that; see, you have taken; so, let us say this is; so, this is t_1, t_2, t_3, t_4, whatever. So, the number of arrivals in this interval which is $\tilde{N}(t_1, t_2)$ is independent of the number of arrivals in $\tilde{N}(t_2, t_3)$ or $\tilde{N}(t_3, t_4)$.

Any non-overlapping intervals, you will look at the number of arrivals here and the number of arrivals here. They will be independent random variables, for any non-overlapping intervals. It does not matter how big it is, it does not matter where it is, as long as they are not overlapping. The random variables will be independent. And if the length of two intervals are the same, then the distribution of the number of arrivals will be the same according to SIP, stationary increment property.

So, SIP says the number of arrivals in any interval, depends only on how big that interval is or not where that interval is located. Independent increment property says that, if there are non-overlapping intervals of time, they do not have to be of the same length, they can be of different lengths. If they are non-overlapping, the number of arrivals in this interval and number of arrivals in this interval are independent, they are independent random variables. Is that clear? They are not the same thing, right? They are two very different properties. It just so happens that the Poisson process has both the properties.