Stochastic Modeling and the Theory of Queues
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Lecture-79
The Jackson Networks-Part 2

So, that main tool of attack is, so what is the goal?

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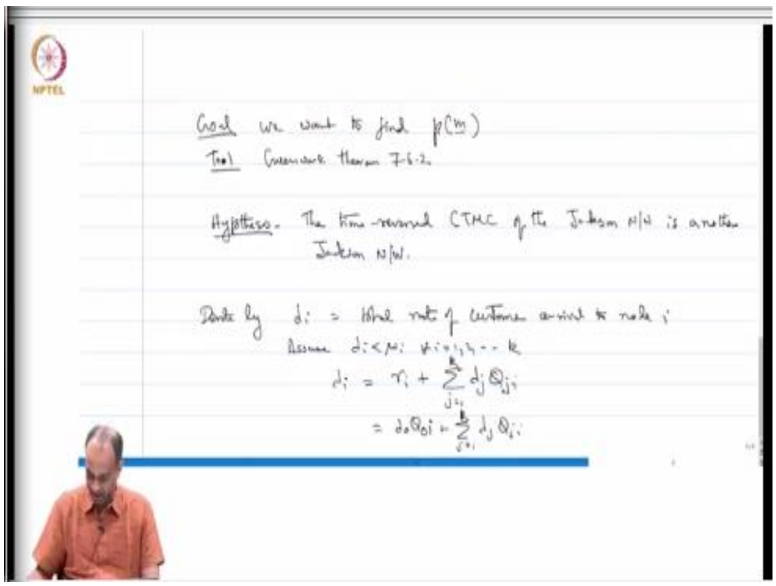

Goal is to we want to find p m for the Jackson network. So, the main tool is going to be a guesswork theorem. The guesswork theorem in particular it is the guesswork theorem 7.6.2 in your book which we studied if you manage to guess the reverse transition rates which satisfies P i Q ij = P j Q ji star, then you can guess the steady state probabilities P i's and that is the correct answer is what your theorem 7.6.2 said.

So, we will use this technique. The hypothesis or the guesswork, the time reversed CTMC of the Jackson network is another but different Jackson network, this is the hypothesis. I am going to guess some transition rates for the reversed action network. So, let me first put down some notation.

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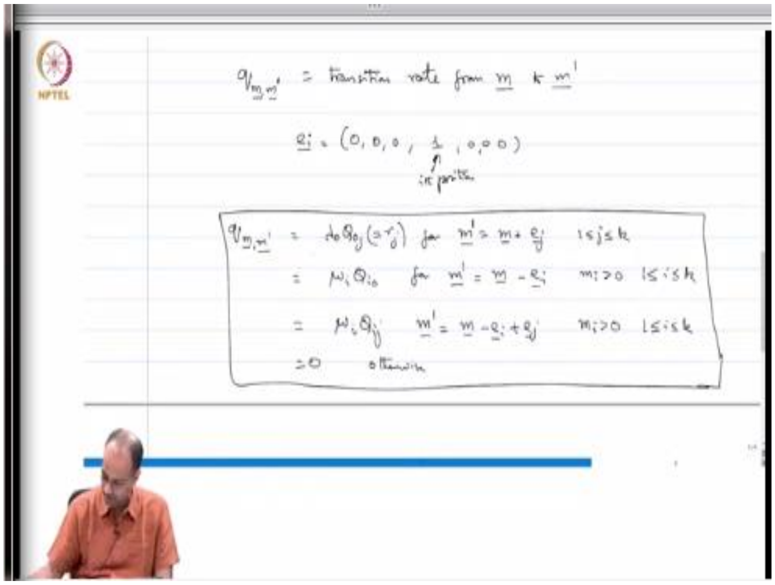

So, this is all this lambda 0 is the echon at arrival rate. So, let me denote, so denote by lambda i as the total rate of customer arrival to node i, which is see lambda i is not the same as r i, r i is the rate of end exogenous arrivals to node i. The total rates of customer arrival to node i will be r i + all the rates of traffic, all the traffic that is coming from other nodes to node i because you can complete service and node j and come to node i also. So, this lambda i will be equal to r i + sum over all the traffic leaving node j.

So, the total rate at which traffic enters node j is also the rate at which it is leaving node j assuming that everything is stable. Of course we have to assume here that, assume lambda, so lambda is total are able to node i. So, assume lambda i less than mu i without that you will not have even the stability of the node i, so this is true for all i = 1, 2 dot, dot, dot, k, then you will have Q ji, so this is the rate at which traffic comes into node i. I can write this as lambda 0 Q 0 i + sum over j = 1 to k lambda j Q ji.

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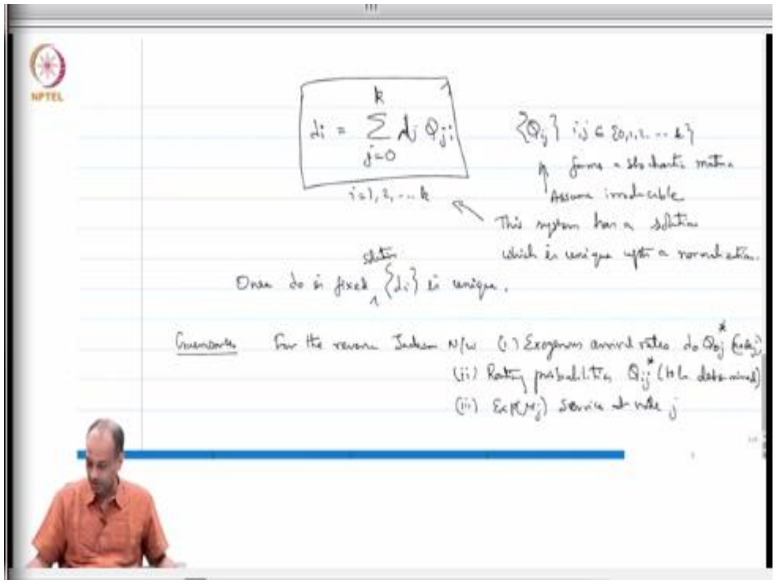

So, in effect what is lambda i? The lambda i satisfy the following equation, they satisfy some over j = 0 to k, so I am just including node 0 also in the sum, lambda j Q ji. So, lambda i which is the total arrival rate at node i satisfies this. See this lambda i the arrival process at Q i may not be a Poisson process. This lambda i is the rate of the arrivals but the arrival process may not be a Poisson process, why?

Because if you have in cases where there is feedback, we even saw with the one queue with the feedback the arrival process seen by the queue can be quite bursty, it need not be a Poisson process. If you have an exclusively feed forward Jackson network like tandem queues or you get out of a tandem queue, split, feed forward into more queues and go out all together without any feedback whatsoever.

Then you will have Poisson inputs to all the queues but the moment you have some feedback, some cycle back into the same queue or some such thing you may not have a Poisson process of arrivals to each queue. So, what is the bottom line? So, a Jackson network at the arrival process seen by a single queue in a Jackson network may not be a Poisson process, it has some rate lambda i but this process lambda i may not be Poisson.

If there is no feedback if the entire Jackson network is feed forward, then it will be a Poisson process. But what is interesting is that even though this lambda i arrival process is not a Poisson process each queue will behave as though it is an M/M/1 queue that is what we are going to say eventually. At any given time t if you take a photograph of the system at time t the occupancy distribution looks like there are independent M/M/1 queues, that is where we are heading.

So, this is the equation satisfied by the lambda i’s. Now this Q ij’s or Q ji’s or whatever they are it is a stochastic matrix. So, this Q ij, if you look at i, j in 0, 1, 2, dot, dot, dot k this forms a stochastic matrix. So, if you have, if it is irreducible, assume that this is an irreducible stochastic matrix. If you assume an irreducible stochastic matrix of this Q ij which are the routing probabilities.

Then this system of equations, when you run this over i = 1, 2 dot, dot, dot, k, so when you run this lambda i over all this k this system has a solution. If you assume an irreducible Q ij this system has a solution which is unique up to a normalization. Now see you can view these lambda i’s are at some level it is like the pi i system for if you think of some DTMC with these transition probabilities Q ij’s and it is an irreducible DTMC.

It will have some pi i and this pi i is, in a DTMC is normalized to 1. Here the lambda is not normalized to 1, these lambda i's are determined the only thing that we know in this lambda i’s is that lambda 0 is known. So, once lambda 0 is known and fixed this is a unique solution to the system of equations, lambda i is unique, the solution lambda is unique. So, that system of equations has a unique solution.

That is just determined by solving the steady state vector of the stochastic matrix Q ij. Now guesswork, so this is all set up so I just define lambda i is the total arrival rate and node i, I said that it satisfies this balance equations which has a solution. The solution is unique, if you fix lambda 0 which is the exogenous arrival rate. Now guesswork, the guesswork is guided by our hypothesis which you have not yet shown but we will show it.

The hypothesis is that the time reverse CTMC of the Jackson network is another Jackson network. The guesswork it is the reverse process is just another Jackson network and I have to specify all the arrival rates. What is the Jackson network specified by? It is specified by the exogenous arrival rates, the server rates and the transition probabilities. Now what would you guess? Exogenous arrival rates for the reverse, so for the reverse Jackson network I do not know already that it is a reverse Jackson network.

But I am going to hypothesize this and propose some exogenous arrival rate, service rates and transition probabilities or the routing probabilities. Exogenous arrival rates are I am going to say are lambda 0 Q 0j star for node j. Number 2, routing probabilities Q ij star to be determined in reverse time, see the same exponential mu j service at node j. So, for the reverse Jackson network I am proposing these as the, this is arrival rate, service rate and routing probabilities.

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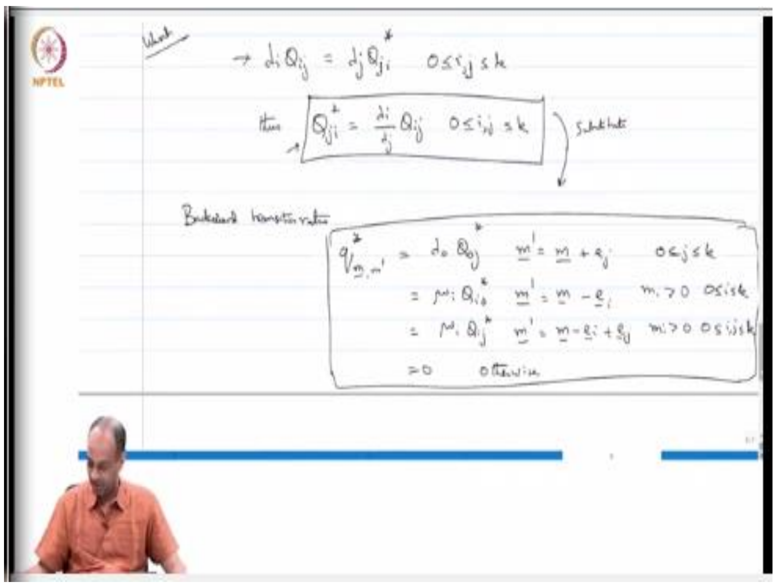

Where I am going to say that Q ij star, I want it to satisfy, this is what I want. lambda i Q ij which is the rate at which traffic goes from node i to node j. I want it to be the rate at which traffic goes from node j to node i in reverse time. The total arrival rate and the total departure rate of node j is still lambda j in both forward time and reverse time. And I want the reverse transition probability Q ji star to satisfy this equation.

This is I want this for both node 0, so I am also defining Q 0j here i, j is not equal to k. So, I am going to have Q ji star = lambda i over lambda j Q ij for 0 less than or equal to I, j is not equal to k. So, this is how I am going to propose the reverse routing probabilities. So, this is some

Jackson network, so I have just come up with exogenous arrival rates of whatever, service rates of mu j and routing probabilities of Q ij star which is like so.

Now I want to show that this is in fact satisfying the guesswork theorems conditions, the guesswork theorem of 7.6.2 or whatever we studied earlier. So, backward transition rates. So, what will the backward transition rates be? So, this is the proposal for the backward Jackson network. So, the backward transition rates will be let us call them q star m, m prime between any 2 states will be lambda 0 it is almost similar to the forward one except you have the stars.

For m prime = m + e j = mu i Q i0 star for m prime = m - e i for m i greater than 0 and equal to mu i Q ij star for m prime = m - e i + e j for m i greater than 0. This is true for 0 less than or equal to i less than or equal to k, 0 less than or equal to i less than or equal to k, equal to 0 otherwise. So, this is the backward transition rates, what I am going to do is substitute this guy into this guy, then what happens? If you just blindly substitute that into this, you can just do this, this is just an algebraic exercise.

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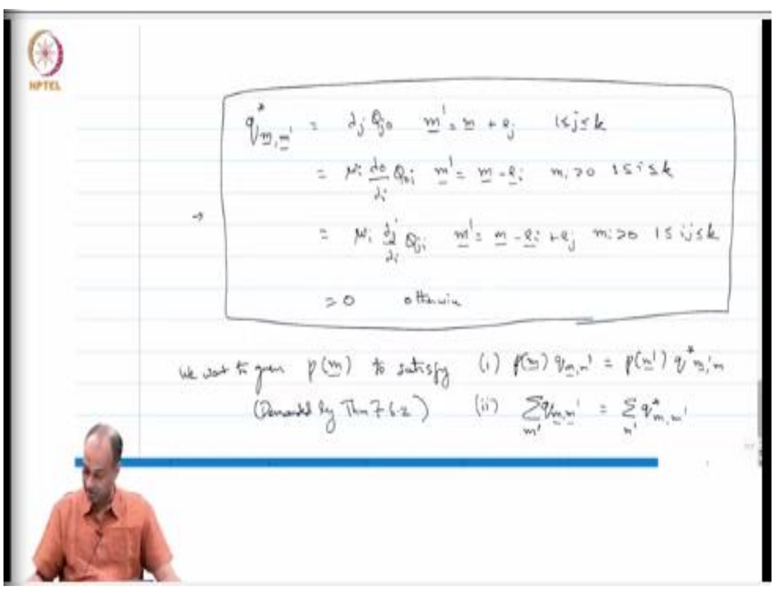

You get Q star, the reverse transition rates m, m prime = lambda j Q j0 for m prime = m + e j = mu i Q i0 star, Q i 0 star I can write it in terms of Q 0i which is mu i, what is Q i 0 star? So, I am going to use that equation. So, I will get mu i lambda 0 by lambda i Q 0i for m prime, so this is just a substitution, m prime = m - e i for m i greater than 0, 1 less than or equal to i less than or equal to k, this is just a blind substitution.

And finally mu i Q ij star will be Q ji lambda j by lambda i Q ji for m prime = m - e i + e j where m i greater than 0 and 1 less than or equal to I, j is not equal to k, also it is I, j = 0 otherwise. So, what have we written? I have written out the reverse transition rates in terms of the parameters of my original Jackson network. See here I had the Q ij stars which are the reverse transition probabilities but I am enforcing some kind of reversibility by using this equation.

I am not enforcing reversibility, I am just looking at what the reverse transition probability should be, it is not reversible of course, I made a mistake in saying that. So, you get q m, m prime star in terms of the parameters of your original Jackson networks. So, this is what I got. Now I want to verify what? Two things, one thing I want to verify is or want to verify meaning what want to, so guesswork.

So, I should do guesswork, I have done guesswork for the backward transition rates, I have to do guesswork for, want to guess p m to satisfy 2 things for the guesswork theorem. One is I want it to satisfy p m q m, m prime = p m prime q star m prime, m, where q stars are the proposed reverse transition rates here, q stars are given by that set of equations. And second I should also guess that I mean I should also satisfy that sum over q m, m prime sum over all m should be equal to sum over q star m, m prime when you sum over all.

So, I think there may be a small mistake here, let me just verify this. Yeah, so I want to verify sum over q, so it should be over m prime q star m prime, so that is what it is. So, these are demanded the guesswork theorem 7.6.2. If I do manage to verify these then I can immediately assert that q star that I have above are in fact the reverse transition rates of the reverse Jackson network.

And that p m is in fact the steady state probabilities of the Jackson network of both forward and reverse Jackson network. So, I have to verify these 2 equations, I have to find p m to satisfy these 2 equations, I already have a proposal for q star, so I want to have a proposal for p m. So, if you look at this, so how do I go about calculating p m is the question now. So, if I just look at this equation.

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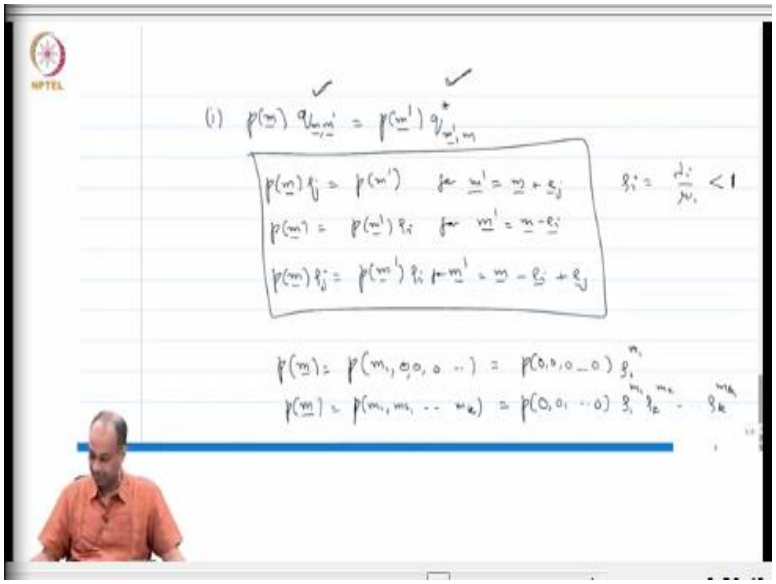

So, that I am just rewriting this first equation p m times q m, m prime, so this is what I want to enforce is equal to p m prime times q star m prime m. Now I can write down so q m, m prime I already know from here. So, this I already know, so that stick and then I know q star from here. So, if I just plug that q and q star from those 2 equations above, what I get is that I will get so p m times rho j = p of m prime for m prime = m + e j.

I will get p of m = p m prime times rho i for m prime = m - e i and I will get finally p m times rho j = p m prime times rho i for m prime = m - e i + e j. Here rho i is equal to just lambda i over mu i, which is less than 1. So, rho i is the load on the ith node, it is the total arrival rate by total service rate. So, this is what I get, this is just algebra, you have this equation 1 which I want to enforce for the guesswork theorem to hold I just I know q m, m prime and q star m, m prime proposal I already have and I just plug it in, I get that stuff.

Now from here it is quite easy. So, what are we looking at? So, if you look at p m prime and look at p m where there is an entry that is decreasing by 1 in the ith position, I have to multiply by rho i. And likewise if the jth increase is increasing by 1, I have to divide by rho j. And if 1 is decreasing by the ith position is decreasing by 1 and the jth position is increasing by 1 then I have to multiply by rho i and divide by rho j.

So, you can start at some state, so if you are looking at a state p m which is equal to p let us say m 1, 0, 0, 0 I can get this by just starting at p 0, 0, all 0 state and then multiplying by rho 1 power m 1, that is because of the above equation, so this is one kind of an m. So, I can just put out, I can just do it again and again so if I have a state which is just p m which is p m 1, m 2 dot, dot, dot m k. I can just obtain it as p 0, 0, 0 rho 1 power m 1, rho 2 power m 2 dot, dot, dot times rho k power m k. I can keep iteratively doing this and get this.

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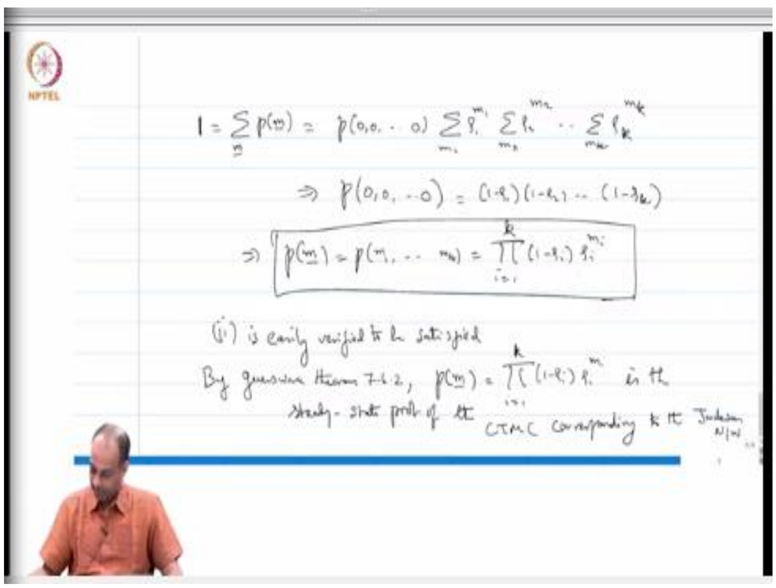

So, I just need to find p 0, 0, 0 which I can do by normalization which is obtained by sum over p m all the states m has to be equal to 1 which means that p 0, 0 comes out. And then I have sum over m 1 rho 1 to the m 1, sum over rho 2 to the m 2, sum over m k rho 2 to the m k. Which means that I can just sum all these geometric series, since all these rhos are less than 1 p 0, 0, 0 will turn out to be just 1 - rho 1 times 1 - rho 2 dot, dot, dot 1 - rho k.

From which I can get p of m which is p of m 1, m k is equal to product i = 1 to k 1 - rho i, rho i to the m i. So, this is the p m that I obtained by enforcing this guesswork theorems condition, which is p m, m prime. I am talking about this equation p m times q m, m prime = p m prime times q star m prime, m, where q star I have already guessed. So, if I enforce this, this is the p m I get.

And of course, so this p m that I have worked out satisfies that equation because I have enforced it. Then the other thing to verify is that sum over q m, m prime = sum over q star m, m prime, this is just an algebraic exercise. Because I have a proposal for both q m, m prime and q star m, m prime. So, q m, m prime I have already know and q star m, m prime I have a proposal.

And this I can just verify by algebra, it works out to be correct. So, I have verified, so what I have done is that I found a p m which satisfies equation 1 and I found p m and q star which satisfy equation 1 and equation 2. And since all the mu's are bounded the technical condition sum over p j nu j is finite is also satisfied. So, by guesswork theorem, so this p m satisfies 1.

Then 2 which is easily verified to be satisfied, just algebra, by 2 I mean, so that equation, that guy. So, this means that guesswork theorem can be used by guesswork theorem 7.6.2 in Gallagher's book, p m = all that pi over i = 1 to k 1 - rho i rho i to the m i is the steady state probability of the CTMC corresponding to the direct Jackson network. And of course q m, m prime star correspond to the reverse transition rates.

So, our guesswork for the q ji stars are in fact correct. The system is positively current and you have found out the transition rates for the reverse process and the steady state probabilities. So, this is excellent, so what is remarkable about Jackson network is that? The steady state probability looks as though it is an M/M/1 queue with load rho i. So, this is a product form of k M/M/1 queues, 1 - rho i times rho i to the m i.

It looks as though, so the steady state probability of the Jackson network looks as though it is a product of independent M/M/1 queues although they are not at all independent M/M/1 queues. See it is not at all the case that the queues are independent, so if you look at q 1 at time 1 and q 2 at time 1.5, they will not be independent, they will be dependent actually because there is all this traffic growing between them.

But at any given time in steady state they look like M/M/1 queues, this is in spite of the fact that their dependent queues and very importantly the arrival process at any given queue need not be a Poisson process as we argue. The lambda i arrival process, the aggregate arrival process at q i is not Poisson in general, it is Poisson only for feed forward networks. So, the moment you have some cycling back, you will not have Poisson.

But it still looks like an M/M/1 queue with load rho i, it looks as though it is an M/M/1 queue with arrival rate lambda i and service rate mu i although the arrival is not Poisson and the queuing processes are not independent. But the steady state corresponds to this product form. So, this is very nice, this is very nice way of analyzing this complicated interconnected queuing systems using Jackson networks.

So, what we have made heavy use of is this guesswork theorem. The guesswork theorem turned out to be very powerful. In fact using this guesswork theorem you can study even more complicated networks. For example you can have state dependent service rates, we have assumed mu i is constant for that server. But you can have server number 3 which serves at different rates whenever there are different number of customers, all that is allowed and not much will change.

In fact there is a book by Frank Kelly which deals with these reversible and product form networks in great detail. But the crux of the argument is just like this, where you can just understand in 1 lecture but there is actually an entire book worth of material on this sort of product form networks by Frank Kelly which is a very nice book. So, I will stop here. So, that concludes our discussion on continuous time Markov processes. We will just have a discussion on semi Markov processes and then we will complete the course, thank you.