Stochastic Modeling and the Theory of Queues
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Lecture-76
Burke's Theorem and the Tandem Queues-Part 1

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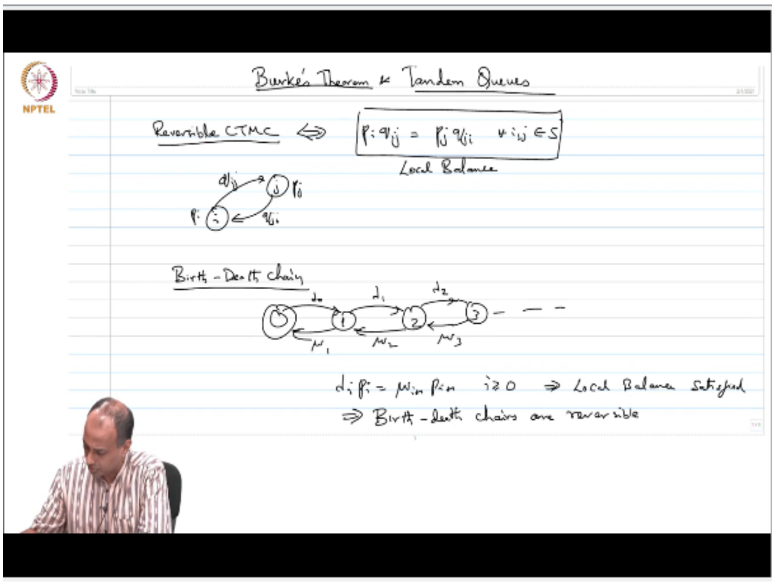

Welcome back, in the last lecture we discussed reversibility of a CTMC. To recall reversible CTMC is just saying that for all ij P i q ij = P j q ji for all ij. So, for every pair of states if you have the rate of transition times the probability of being in state i is equal to the rate of transition from j to i of times the probability of being in state j. If this holds for every pair of states ij then we know that the CTMC is reversible we derived this actually.

This is equivalent to saying that for at least for nice CTMCs this reversibility of the CTMC is equivalent to reversibility of the embedded DTMC this we already saw. Now this is what is called local balance, so for any 2 pairs of states you have this, this is called local balance. We saw that the satisfaction of the local balance equation automatically implies the satisfaction of the global balance equation.

We also derived 2 guesswork theorems, so if you do happen to find some P i's that satisfy this then those P i's are automatically the steady state probabilities and the chain is reversible. And likewise we also had another guesswork theorem in which you guess both the P i's and the q ij stars and then simultaneously you know the steady state probabilities and the transition rates of the reverse CTMC. So, those things we discussed earlier. Now if you look at a Birth-Death chain, so it looks like this, you may recall that, now the transition rates are like this, so you will have lambda 2 mu 2, lambda 0 mu 1, lambda 1 mu 3 and so on.

So, in this chain we already know that the balance equations if you explicitly write out the global balance equations you will actually get lambda i times P i = mu i + 1 P i + 1 for i greater than or equal to 0. So, this automatically satisfies local balance, so we should write down the global balance from which we could recover this. So, this implies local balance satisfied, this implies Birth-Death chains are reversible.

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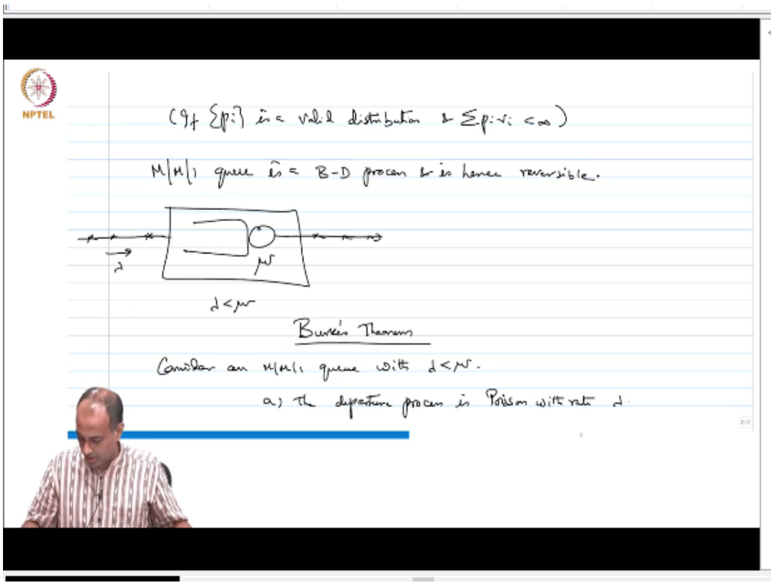

Assuming of course that if P i is a valid distribution and the sum over P i nu i is finite if it is not a chain which has some irregularity properties. Then Birth-Death chains are automatically reversible, which means that any Birth-Death process or maybe I should say instead of Birth-Death chain I should say Birth-Death CTMC or a Birth-Death process. Any Birth-Death process is reversible; it looks statistically identical in forward time as well as reverse time.

Therefore I mean the M/M/1 queue is a Birth-Death process and is hence reversible. So, if you have an M/M/1 queue which is stable say you have a poisson process of rate lambda coming into the queue and you have customers departing. So, consider an M/M/1 queue in which they have poisson arrivals of rate lambda and some server of rate mu exponential service with lambda less than mu.

This is a Birth-Death process and is hence reversible. So, if I take a video tape of this process if this customers joining this queue getting served and leaving if I had to just play it in reverse the process would look statistically indistinguishable from the forward process. So, the process in backward time and the forward time would look statistically exactly the same. Notice that the process of departures in forward time will be the process of arrivals and reverse time.

So, the process in reverse time is a M/M/1 queue, so its arrival process in the reverse time must be a poisson process of rate lambda. So, we can conclude that the departure process is in fact a poisson process of rate lambda in an M/M/1 queue, this is Burke's theorem which we did in the time sample version but we can do it in the full CTMC version now. Now that we know reversibility of CTMCs we can actually state burke's theorem in continuous time and we will also discover that it has one more feature which is not easily discoverable in discrete time. So, let us do Burke’s theorem. Consider an M/M/1 queue with lambda less than mu then we can make 3 statements, a the departure process is a poisson process with rate lambda.

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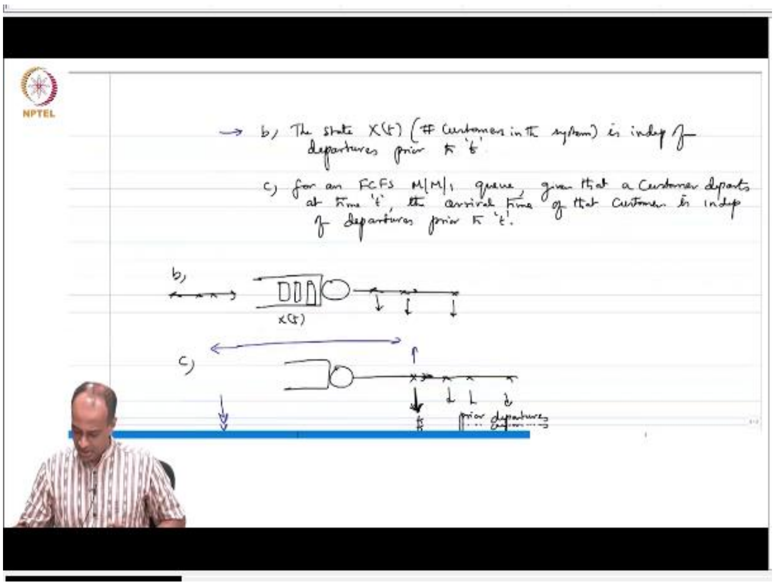

B, the state X t which is the number of customers in the system is independent of departures prior to t and 3. So, the first 2 parts we already sort of know this because we did Burke's theorem in DTMCs for the time sample Markov chain. Remember you had DTMC with lambda delta, mu delta, transition probabilities viewed as a DTMC that was a reversible DTMC.

And we could infer these properties in discrete time. Now we are doing the full continuous time version. We are also going to get a a and b were already known to us, it is sort of a repeat part C is nu it comes out only in the continuous time version. Part C for FCFS a first come first serve M/M/1 queue given that customer departs at time t, the arrival time of that customer is independent of departures prior to t.

So, part c is unique to continuous time, it says that and it holds only for first come first serve M/M/1 queue. It is not the first 2 parts do not assume anything about the service discipline part c is about first come first of M/M/1 queue. So, given that a customer departs at a particular time t we are saying that that customer's arrival time let us say this is t and the customer departs at time t the arrival time of the customer is independent of departures prior to t.

And this is really not true in discrete time because you have these small slots which prevent departures and arrivals happening at the same time. So, if a customer departs at a particular time slot in the time sample chain you can say that the customer did not arrive in the time slot. So, you cannot have independence in the time sample DTMC but in CTMC you have this independence this is this added nice feature that comes in the CTMC.

So, just to recap so a the departure process is a poisson process of rate lambda. That is because if you look at the reverse process the departure process in the forward chain the forward queuing system is the arrival process in the reverse queuing system. And we know that the reverse queuing system is also an M/M/1 cube with poisson arrivals of rate lambda. Therefore the departures from an M/M/1 queue will be poison over rate lambda.

It should not surprise you that the departure process is of rate lambda; it should surprise you that it is a poisson process because if customers are coming in to a queue at a rate lambda and if the queue is stable we have assumed lambda less than mu. They have to leave at a rate lambda because there is no accumulation of customers inside the queue. So, you should not be too surprised that it is a poisson process of rate lambda, the surprising part is that it is a poisson process, the surprising part it is not that it is of rate lambda, some people get confused here because students think often that the departure process should be poisson of rate mu.

That is not correct because you have exponential services of duration 1 by mu when the server is busy. When the server is not busy well there are no departures of course, what this part a is saying is that the unconditional departure process is poisson of rate lambda. Of course conditioned on the queue being occupied the departure process will be of rate mu. So, that is not too surprising, but that the departure process is poisson of rate lambda unconditionally is the non-trivial consequence of Burke’s theorem of reversibility essentially.

And this result is not easy to prove at all if you do not use reversibility arguments. If you go from elementary principles it could be quite difficult to prove, I do not know of a simple proof of proving that the departure process from an M/M/1 queue is poisson of rate lambda without using reversibility, I do not think it is easy. Part b it says that the state X t which is the number of customers in the system is independent of departures prior to t.

So, if you look at some time t let us say so part a is easy. Now if you look at part b just to reason it out, so this is your system, so you have poisson arrivals and some departures. So, if you look at time t, so there is some customers in service. Let us say X of t, X of t is the number of customers in service at time t. We are saying that X of t is independent of these guys who have left past departures.

So, intuitively we are saying that if you happen to see a whole bunch of departures in the last 5 minutes you may think that maybe the queue is empty, queue is sort of empty, there may not be too many people in the queue. That is certainly not true for an M/M/1 queue. In fact looking at the departures in the immediate in the past whatever interval of time tells you nothing whatsoever about X of t.

They are independent; to see that this is the case note that these departures prior to t if you play it in reverse they will look like future arrivals in the reverse M/M/1 queue. So, at time t departures prior to t in forward time or arrivals to be coming in future time to the M/M/1 queue and of course X of t is independent of future arrivals in the reverse process. Because the reverse process is also an M/M/1 queue and the current state of the system is dependent only on past arrivals and past service times and it is independent of future arrivals.

Since X of t is independent of future arrivals in the reverse system, the reverse system being an M/M/1 queue the same should also be true in the forward time. Therefore X of t is independent of past departures. So, that is how you argue part b and finally for part c what are we saying in part c? For a first compressor M/M/1 queue given that a customer departs at a time t the arrival time of that customer is independent of departures prior to t, so same picture. So, let us say that this is my tagged customer who left at time t. And then they were, so this is a departure at time t, these were prior departures.

So, meaning that this departure happened at time t and these departures happened before t, we are saying that the arrival time of this customer is independent of prior departures. So, the arrival time of this particular customer, this guy who left is dependent on his own service time and this is now FCFS. So, a customer waits only for those customers who came ahead of him or her and does not wait for anybody who comes after.

So, my total system time, so, if I am this customer departing at time t my total system time is independent of those who come after me. So, if I come in at a particular time and I leave at time t, so the total time I spent in the system is independent of arrivals who come after me. So, if you play time in reverse, arrivals that come after my arrival will translate in reverse time to departures prior to t.

So, as though in reverse time so if this is all forward time, so if I draw reverse time in a different colour. So, this guy came to the system at some point and the arrival time so of this customer his arrival time at some point here would seem like a departure in reverse time. Now so this total system time will be independent of future arrivals in the reverse system. But the future arrivals in the reverse system are prior departures in the forward system.

This is because now we are crucially using the FCFS property therefore you can argue that part c holds, just I am not writing it out very precisely but this is the essential argument. In FCFS I only wait for those who have come ahead of me, I do not so my system time is independent of arrivals that come after I have arrived. So, if you just reverse time part c follows.

But of course this is not necessarily true if there is some other service discipline, because if I am you doing some kind of last compressor or processor sharing or whatever the time I spend in the system could depend on people who come after me because they could preempt me or they could take some of my service or whatever. So, in that case part c will not hold, I hope this is correct, I hope this is clear.

Now please note that there is nothing in this Burke's theorem at least the first 2 parts. If you look at part a and part b, part a and part b do not in any way depend on the service rate being mu in all states. So, I could have as long as my arrivals are poisson I could have whatever service rate mu i in state i that I want.

So, I can have different rates of service mu i in state i, so when there are 16 customers in the system I could say a server at a rate nu 16, when there are 20 customers in the system I could serve at some other rate nu 20 and then when there are 17 customers in the system I could just refuse to serve nu 17 can be 0. All that matters is as long as my mu's are such that the system is positive recurrent.

Then it is being a Birth-Death chain it is still reversible. So, even a queue in which the service times are totally heterogeneous across states you will have Burke's theorem holding. The departure process will still be poisoned with rate lambda which is very remarkable. No matter what service rates you employ in various states as long as the system is positive recurrent it will be reversible and the departure process will be poisson of rate lambda and part b will also hold for the same reason.

Because you have the state at any time will be independent of departures prior to t because in reverse time the state at any time is independent of arrivals in the future. These 2 parts continue to hold. So, even in an M/M/s system where there are s servers you have parts a and b holding, part c may not really hold because if you have let us say an M/M/s server if you have a first first-come first-served M/M/s server it is not really the case that people who come after me do not it is not true that people who come after me cannot depart before me because they could go to a different server and depart before me.

So, this argument does not hold for M/M/s queue. So, it is a subtle point that part c continues to hold for an M/M/s queue but not for the reason that I mentioned, it is still true but not the proof argument is more subtle and I do not want to get into it. But part a and b hold regardless of any service rate you employ in any state which is a very remarkable fact of this

Markovian queuing systems you do not even need M/M/1. Burke's theorem is stated always for M/M/1 but you know it is much more true much more generally.

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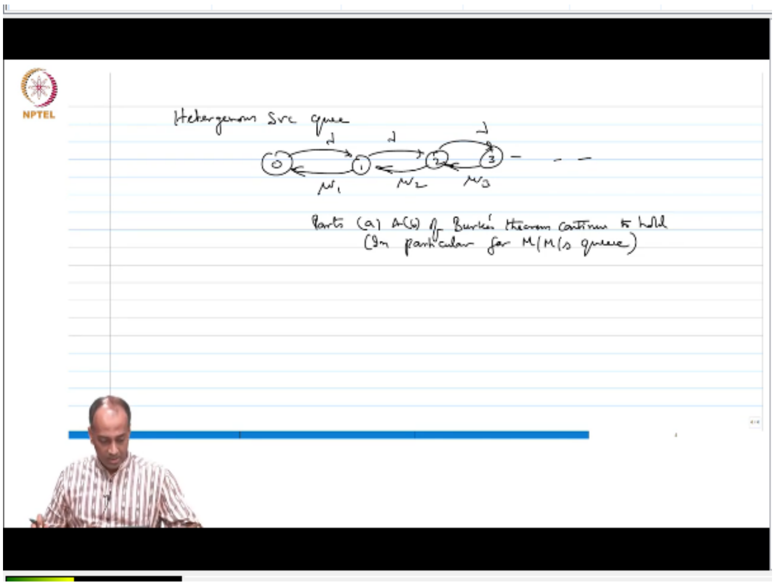

You could have a queue in which you could have a heterogeneous service queue, so you could have a poisson arrival. So, you could have lambda arrivals rate lambda in all states but the service rates could be whatever you want mu 1, mu 2, mu 3 as long as the system is positive recurrent parts a and b of Burke's theorem continue to hold in particular for M/M/s queue.

Because M/M/s queue just has mu 2, mu 3, mu on till s mu for service rates and then continues that s mu which we have already seen. So, parts a and b continue to hold for any heterogeneous service queue. This is a pretty remarkable result and neither of this is easy to prove without reversibility.