Stochastic Modeling and the Theory of Queues
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Lecture-75
The Reversibility of Continuous Time Markov Chains

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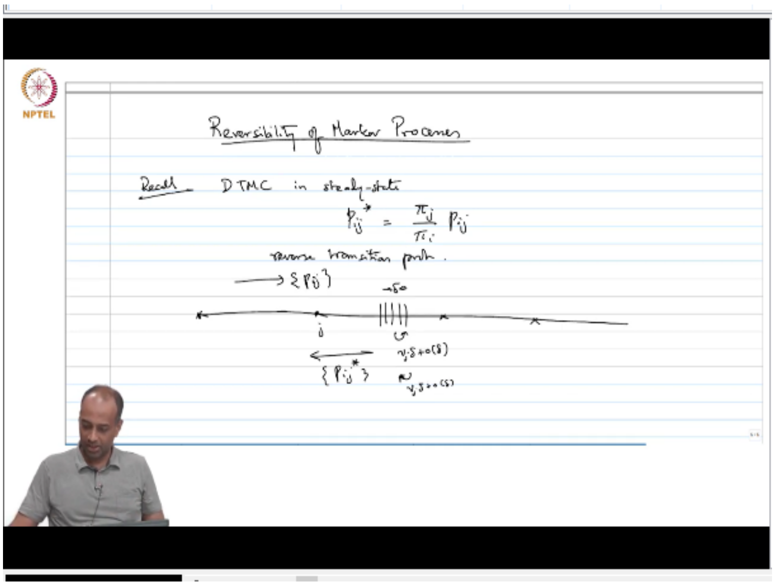

Welcome back, now we will study reversibility of markov processes or reversibility of CTMCs. Recall that for a DTMC in steady state that is if you start of DTMC in pi i then P ij star which is basically pi j over pi i P ij is the reverse transition probability. This we studied in great detail. So, just to refresh your memory we said that for a DTMC if you have a DTMC running in forward time the reverse time is always a markov chain but it could be a inhomogeneous markov chain.

However, if you run the forward markov chain in steady state that is you start in pi i or you let the markov chain run for a very long time so that it reaches pi i. So, either way if the forward running chain is in steady state the reverse chain is also a homogeneous markov chain, it is a homogeneous markov chain with transition probability P ij star given by this expression on your screen.

Now we are going to assume that the embedded DTMC is in steady state; embedded DTMC has transition probabilities P ij, so the reverse embedded is the reverse process of the embedded DTMC will have reverse transition probabilities P ij star. But in the CTMC world you are also having these holding times. So, just to show you a picture what happens is let us not put 0 because let us say assume that we are in steady state.

So, here are all the state transitions, so there are transitions in the forward time these are governed by P ij's. In reverse time these transitions are governed by P ij star the transition probabilities between the states at the transition instances. So, let us say you are in some state j. So, if you look at these little, little delta intervals so the holding time in each of these intervals between successive transitions is exponential with parameter nu j given that you are in state j.

So, the transition probability that you leave stage j from this slot to this slot will be nu j delta plus o delta in forward time. So, even in reverse time it is true, the probability of going from so it is after all exponential random variable the probability of making a transition out of j in reverse time is also nu j delta plus o delta. So, what happens is that when you look at it in forward time you have this P ij transition probabilities and exponential holding times of rate nu j's whenever you are in stage j.

When time is running in reverse you have P ij star transition probabilities with the same exponential holding times. So, the reverse process is also a markov process, because you have it, but the embedded markov chain is different that is all that we are saying. So, when the markov process runs in forward direction the embedded transition probabilities are P ij and the holding times are exponential nu j whenever you are in state j.

In reverse time the embedded DTMC has transition probabilities P ij star which is given on your screen and holding times continue to be the nu j's corresponding to those states. And of course just like in the discrete world the P j which is pi j over nu j divided by sum over k pi k over nu k will remain the same for the forward process as well as the reverse process.

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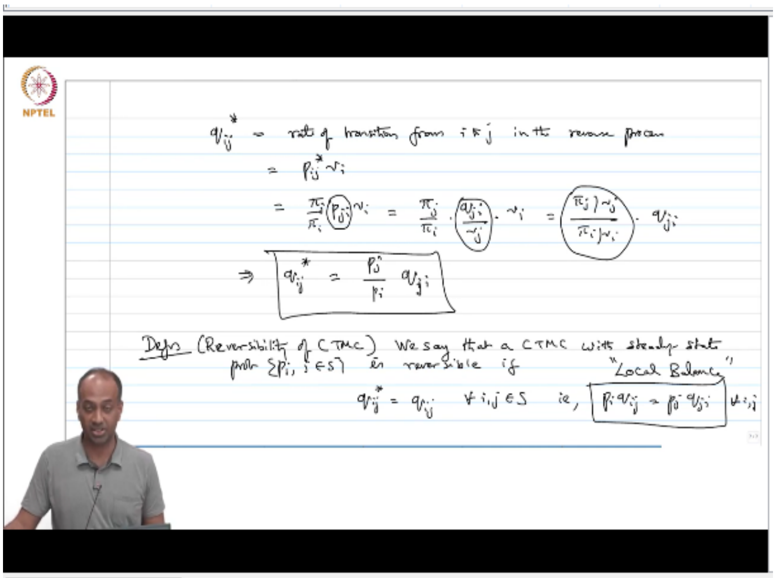

Now what about the reverse rates, if you look at q ij star which is the rate of transitions from i to j in the reverse process. This is just P ij star times nu i because you are going from i to j. P ij start times nu i. You are holding for nu i duration and then jumping to some state j. So, if you write this out P ij star from the earlier discussion P ij star we know to be equal to that. So, if you just put that expression into this you get pi j over pi i. I think I made a mistake P j this should be P ji sorry I apologize so this should be P ji P ji nu i**.**

So, these are your transition rates in the reverse process. So, it is pi j over pi i. Let me just check that this is correct, it is nu j, nu i pi j P ji over pi i. So, this I can write this again as if you just look at this as pi j over pi i, I am just going to rewrite this q ji over nu j times nu i. So, what have I done now? I have just replaced P ji with q ji over nu j. This is a just very basic algebra. So, what does this work out to be? This is just pi j over nu j divided by pi i over nu i times q ij.

So and what is pi j over nu j divided by pi i over nu i is just P j over P i because we know that P j is proportional to pi j over nu j. That we already know, so what does this imply after these manipulations you get q ij star = P j over P i q ji. So, this is q ji. Now this is correct. So, now this is nice P i q ij star = P j q ji. So, q ji star which represents the reverse transition rates is given by q ij star = P j over P i q ji.

So, what are we saying, so if you are given a forward CTMC with transition rates q ij which completely specified the process and assuming that this is a nice CTMC with the steady state pi and all that. The reverse process is in fact a CTMC whose transition rates are q ij star = P j over P i q ji. Now definition, now we come to the important definition of reversibility of a CTMC.

So, just like in the DTMC world, so remember that if a forward steady state DTMC is a forward thing is a markov chain the reverse chain is also a markov chain which transition probability is P ij star, but that is not what reversibility is. The reversibility means that the reverse markov chain and the forward markov chain have the same transition probability matrix. So, in the same spirit we will define CTMC to be reversible, CTMC with steady state probabilities P j is reversible if q ij star = q ij for all ij in S. ie P i q ij = P j q ji for all ij.

So, this is the definition. You say that CTMC is reversible if the forward transition rates and the reverse transition rates q ij star = q ij for every ij. So, between any 2 pair of states the transition rates in the forward process and the transition rates in the reverse process must be equal which leads to the nice equation this local balance kind of an equation P i q ij = P j q ji. So, you can view this as some kind of a local balance in the CTMC world. If this is the case then the process is reversible this definition.

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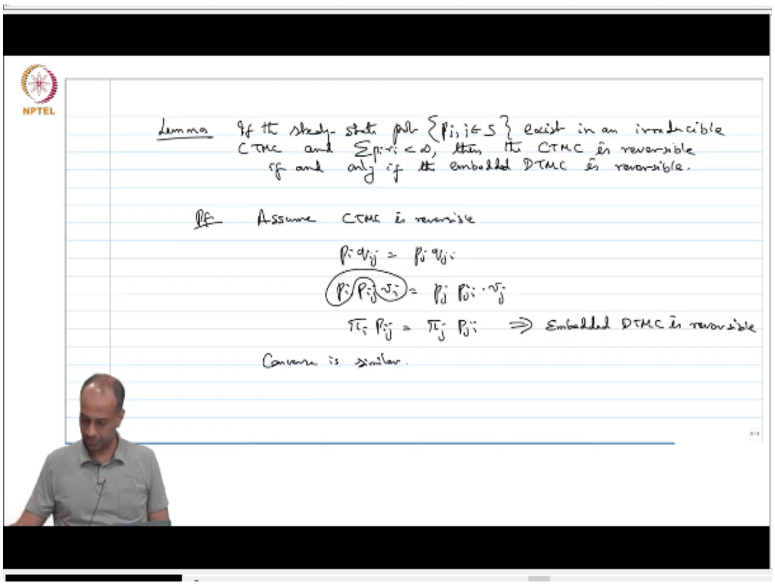

Now we can prove a lemma, now that says that if the study state probabilities P j’s exist in an irreducible CTMC and sum over P i nu i is finite then the CTMC is reversible if and only if the embedded chain is reversible, embedded DTMC is reversible. So, here we are saying that there is an equivalence between the reversibility of the CTMC and the reversibility of the embedded DTMC.

Reversibility of one implies the other, both directions; of course assuming that this is a nice irreducible CTMC with some more P i nu i less than infinity, you do not have this crazy irregular avalanche process and all that. So, this is very easy to prove; actually if you just look at it so assume the CTMC is reversible then what do you get P i q ij = P j q ji which means that P i q ji is what P ij times q ij's P ij times nu i and q ji is P ji times nu j.

So, we have this. Now we know that so P i nu i in the world when some over P i nu i is finite, if P i take that term that something that is proportional to pi i and likewise the corresponding term here is proportional to pi j. So, you get pi i P ij = pi j P ji. This implies that and of course each of these pi i is positive because some more P i nu i is finite. This implies that embedded DTMC is reversible.

So, one direction is shown if the CTMC is reversible then the embedded DTMC is reversible. The converse is similar. So, this lemma very simply characterizes the reversibility of the CTMC as being equivalent to the reversibility of the embedded DTMC and since we understand DTMC reversibility well. So, there is not so much of a conceptual difference between reversibility of CTMC and reversibility of the embedded DTMC.

They are in fact exactly equivalent. Now just like in the DTMC world see there was this desirable feature of course if you have pi i P ij = pi j P ji for the steady state probabilities pi i's then you know that the DTMC is reversible. But sometimes you may not know the steady state probabilities pi j's**.** So, in the case we had a theorem that said if you manage to find any numbers at all by guesswork or whatever that satisfy these local balance equations.

Then you immediately could assert that these pi j's are in fact the steady state probabilities and that the markov chain is reversible and that the markov chain is positively correct. A similar result holds in the CTMC case. So, even if you do not know the P i's, you may not know the steady state probabilities P i's for a CTMC but if you do manage to find some numbers P i that satisfy P i q ij = P j q ji. Then you can prove that the CTMC is reversible.

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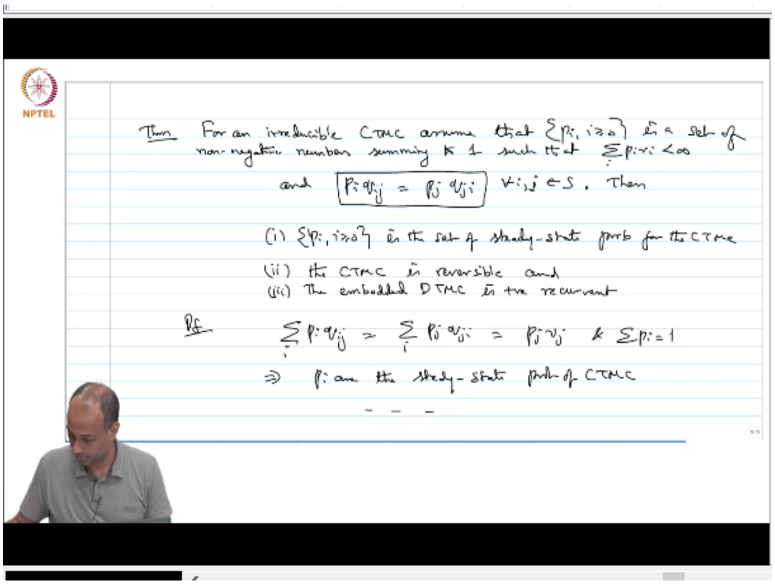

This is a important theorem for an irreducible CTMC assume that P i is a set of non negative numbers summing to 1 such that sum over P i nu i is finite and P i q ij = P j q ji for all ij in S. Then one P i is the set of steady state probabilities for the CTMC. Number 2 the process is reversible. Let me say the CTMC is reversible and the embedded DTMC is positive recurrent. So, this is a very nice theorem. So, if you manage to find somehow guess some numbers P i you do not know the steady state probabilities.

So, you somehow manage to find the numbers which satisfy local balance and such that some over P i nu i is less than infinity and P i sum to 1. Then the P i is that you did guess are in fact the steady state probabilities and reversibility comes for free and the embedded DTMC is positive recurrent. The reason that this is true is if you just look at you sum this local balance equation.

If you have sum over P i q ij is equal to sum over i P j q ji. So, P j comes out over the summation and sum over q ji is what is just P j nu i, this is P j outside, P j nu j. So, this all I am saying is P j comes out some more q ji is nu j. So, I have sum over pi P i q ij = P j nu j. Now these are in fact the balance equations that P i satisfy. And of course some over P i = 1 that you already know.

So, this implies P i's are all positive and are in fact the unique steady state probabilities of the CTMC which we already saw. This we already know. If you find a solution to P i which satisfies the balance equation some over P i q ij = P j nu i. Then those P i’s are in fact the unique solutions of the steady state probabilities of the CTMC. Now P i’s are in fact the steady state probabilities of the CTMC and they satisfy local balance therefore the CTMC must be reversible.

Once you know now that P i's are the steady state probabilities of the CTMC and that local balance is satisfied therefore the CTMC is reversible. And now we since sum over P i nu i is less than infinity you can also get your pi i's to be something strictly positive. Meaning that the embedded DTMC is positive recurrent. So, those steps you can complete, but this is a very useful result, this is a very nice result to have.

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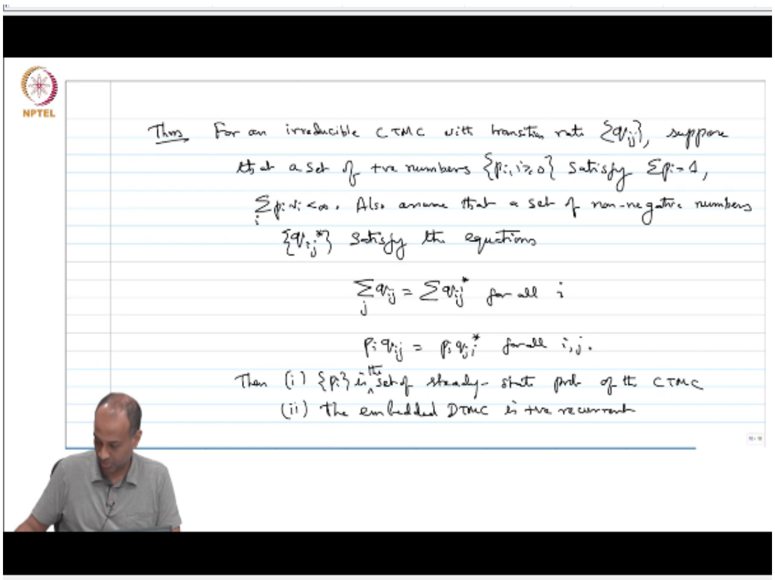

Finally there is a extension of this theorem which says that if you somehow manage to guess numbers q ij star which satisfy certain properties then you automatically get the transition rates of the reverse process. This is like a generalization of the theorem that we just saw for an irreducible CDMC with transition rates q ij. Suppose that a set of positive numbers P i satisfy normalization sum over P i nu i finite.

Also assume that a set of non-negative numbers q ij star satisfy the equations for all i, P i q ij = P j q ji star for all i, j. Suppose this q ij star satisfy some more q ij is equal to some more q ij star for all i and P i q ij = P j q ji. star Then one P i is the set of steady state probabilities of the CTMC. Number 2 the embedded DTMC is positive recurrent.

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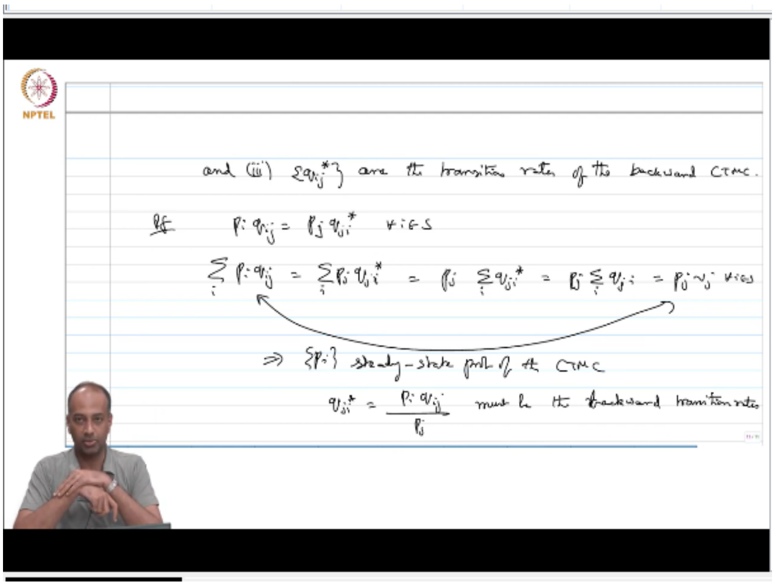

And 3 perhaps most important of all q ij star, whatever you guessed this q ij star are in fact these transition rates of the backward CTMC. So, you have a CTMC running in forward time with rates given by q ij and somehow you manage to guess some numbers P i and some numbers q ij star. You do not know anything about these numbers except that P i’s sum to 1 sum over P i nu i is finite.

Also this q ij star satisfy the 2 equations shown here. Then P i's are in fact steady state probabilities the embedded DTMC is positive recurrent and the q ij stars that you manage to find satisfying these equations are in fact the transition rates of the backward CTMC. Please keep in mind that we are not saying that this CTMC is reversible no, we are just saying that q ij is star I managed to find that satisfy this property.

Then that q ij star is in fact the transition rates of the backward markov chain. So, how do we prove this? If you just look at P i q ij = P j q ji star for all i. Now you can just sum this summing over i, you get sum over i P i q ij is equal to sum over i P j q ji star is not it, they may come with P j q ji star. So, this is a j. Sum over i P j q ji star. That is equal to P j comes out sum over q ji star.

Now sum over q ji star is equal to some over q ij. So, what is sum over q ij nu j. So, this is equal to P j sum over q ji by assumption that is equal to P j nu j. So, sum over P i q ij = P j nu j. So, these are again the balance equation. So, the P i’s that I managed to guess satisfies the global balance equations sum over P i q ij = P j nu j for all i. So, this implies and of course P i’s are all positive and they sum to 1.

This implies P i’s are the unique steady state probabilities, there is a unique solution to this equation and they are the steady state probabilities of the CTMC. Now this P i’s are all strictly positive and sum over P i nu i is finite. So, I can get strictly positive pi i’s from these P i’s which implies that the embedded DTMC is positive recurrent.

And finally now that P i's are the interpretation of the steady state probabilities then q ij star which satisfies so you have q ji star = P i q ij over P j must be the reverse transition rates, backward transition rates but that we know to be the backup transition rates. If you want you can replace P i by that pi i’s are positive. So, you can write this in terms of pi i's and nu i’s and then argue that these are in fact the backward transition rates as we have already done.

This is very useful in guessing somehow that the behaviour of the reverse process, this can be used in for example Jackson networks of queues. A Jackson network of queues is a connected queuing system which we will briefly steady later. But these 2 theorems which are known as guesswork theorems are useful in many contexts. Whenever some educated guest tells you something about either the reversibility of the process in the first theorem or in the second theorem it does not say the process is reversible it just characterizes the reverse transition rates, we can stop here for now.