Stochastic Modeling and the Theory of Queues
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Lecture-73
The Chapman-Kolmogorov Equations for CTMC's

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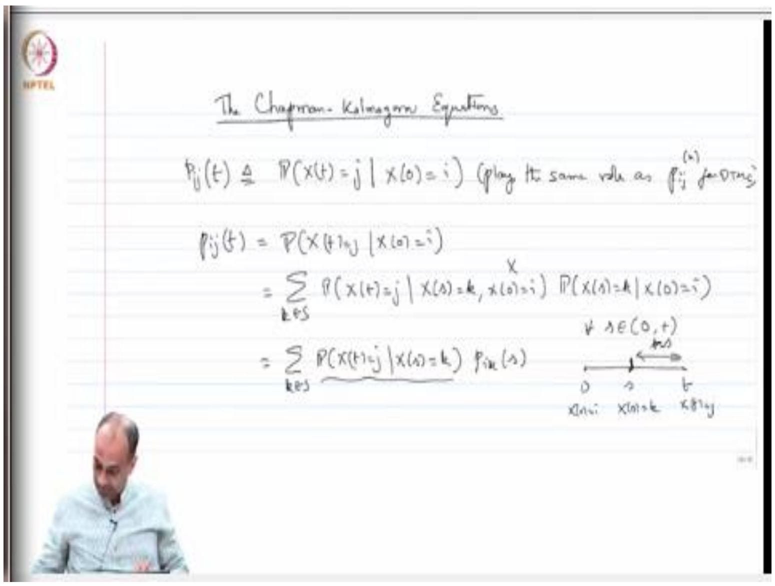

Welcome back, now we will discuss the Chapman-Kolmogorov equations for CTMCs. The Chapman-Kolmogorov equations deal with a quantity P ij of t which is defined as the probability that the process CTMC at time t is in state j, given that the process at time 0 was in state i. So, given that the process started in state i at time 0 what is the probability that time t the process will be in state j? This is P ij of t, so this plays the same role as P ij n for DTMC.

So, P ij n is the probability of being in state j after some time n in a DTMC. Now we are talking about at CTMC. So, in continuous time what is the probability of so being in state j at time t, if I start in state i at time 0. Now the Chapman-Kolmogorov equation relates P ij of t to P jk of S and all that for S less than t as we will see soon. So, I can write P ij of t as follows, probability of given X 0 = i which is by definition which is equal to the probability that X of t = j given let us say X of s = k X of 0 = i.

So, I am basically conditioning on the state at some intermediate time s to be equal to k times probability that X of s equals k given X of 0 equals i and then sum over k all states. So, this is true for all s in 0 t. So, now if you notice this by the Markov property I can drop that conditioning because conditioning on the process being as time s being in state k it does not matter where I started. So, I can write this as sum over k belongs to s P X t = j given X of s = k times what I have here? Probability of X s = k given X 0 = i is nothing but by definition it is P i k of s.

And what is that equal to? This also I can simplify, see if at time s the process is in state k, so the probability of at time t the process being in state j. Now this Markov property guarantees that this is equal to, it is as though the process starts at time 0 in state k. And then you are just looking at a time interval t – s, so just to draw this out. So, looking at this is time 0, this is some time s and that is some time t. So, if X of s = k because X 0 = i, X of t = j, that is what I am looking at. So, at this point if I am at state k, it is as though I am starting a nu process for this time duration t - s. And it is also time restarts at time 0 and I am in state k.

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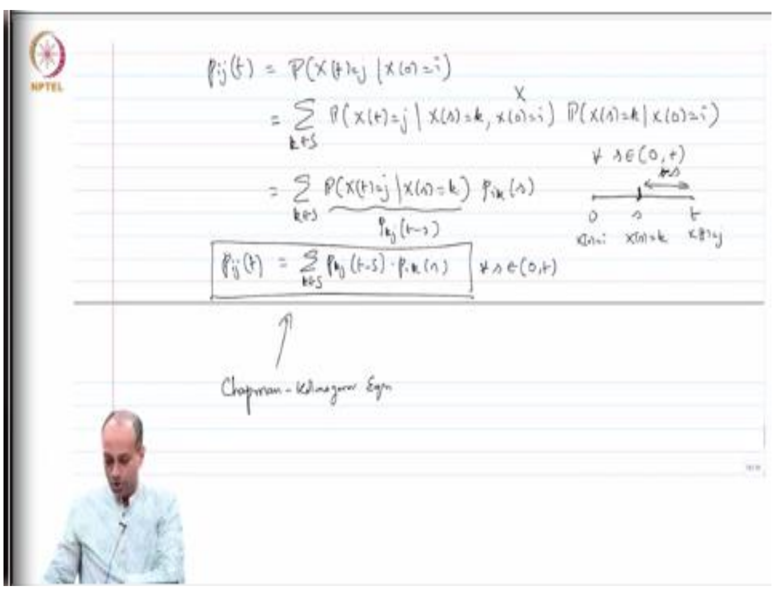

So, by the property of this Markov property this guy just becomes P kj of t – s. So, i get P ij of t = sum over k belongs to s P k j of t - s times P i k of s, this is called the, so this is true for all s n 0 to t. And this guy is called the Chapman-Kolmogorov equation for CTMCs. It basically expresses P ij of t which is the transition probability at time t in terms of transition probabilities at smaller times. And you can choose whatever s you want, s can be anything in 0 to t and this equation is still valid. Now as you can imagine this Chapman-Kolmogorov equation just ends up expressing P ij of t in terms of P kj of t - s and P ik of s which also we do not know really.

So, we have just expressed P ij of t in terms of similar other quantities that we do not seem to know. So, it is not very clear that this is very useful, at least in the discrete case we could inductively solve for P ij of n then find out P ij of n + 1 and all that using the Chapman-Kolmogorov equations. Here it does not seem obvious how we can even solve for this P ij of t? Because what you have on the right hand side is also similar quantities that you do not know.

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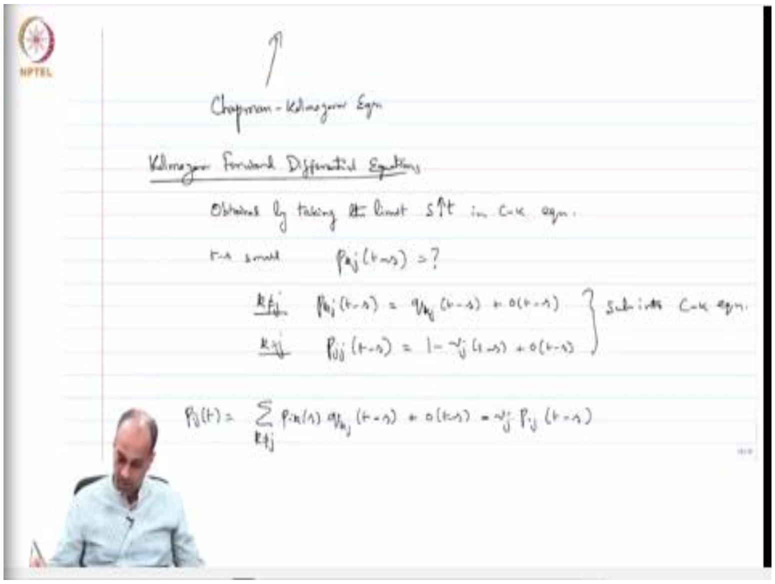

Now the solution to this see in the DTMC case you could just inductively do it, start with n = 2 then go to n = 3 and all that. In the continuous time case, in the CTMC case the Chapman-Kolmogorov equations can be converted to a differential equation form. So, you can just look at very small time increments and look at the evolution of this P ij of t. And then express you can write out a coupled system of ordinary differential equations which can ultimately end up solving for this P ij of t, that is what we will do now.

So, there are 2 types of differential equations one is called the Kolmogorov forward differential equation and the other is called the Kolmogorov backward differential equation. So, the Kolmogorov forward differential equation is obtained by taking this t - s to be really small, so what you do is send s towards t. So, if you look at this picture you send this s make this s approach t, make this s - t really small.

And then you write out the Chapman-Kolmogorov equation for t - s being really small and you get a set of differential equations that is called the Kolmogorov forward differential equations. So, essentially obtained by taking the limit s approaching t from below in the Chapman-Kolmogorov equation, so let us do this. So, I am going to take t - s is really small, so I am going to say let us see what happens to P kj of t – s?

So, I am going to have t - s really small and what is P kj of t - s? Let us first consider the case k not equal to j, if k is not equal to j P kj of t - s is basically the probability of going from state k to state j in a very small interval t - s. So, this should be equal to the rate of transition from k to j multiplied by this very small time interval which is t - s. So, P kj of t - s can be approximated as q kj which is the rate of transition times delta, delta is here is just t - s + o delta which is o t - s. And for k = j, you are just looking at P kk, so it is a probability of self transition.

So, this will just become 1 - nu, this is the probability of self transition which is the probability that none of these exponentials actually fire in the small time interval. So, this will just become, so P k = j, so maybe I should write like this P jj of k = j now, so P jj of t - s = 1 - nu j of t - s + o of t - s. So, you go fix this back into, so sub into Chapman-Kolmogorov equations. So, then what do you get? So, you get P ij of t = sum over k not equal to j P ik of s times q kj times t - s + o t - s. And then you have take the term k = j also, in the case k = j you get - nu j P ij of t - s nu j P ij times t - s I think I made a mistake here, so that is correct.

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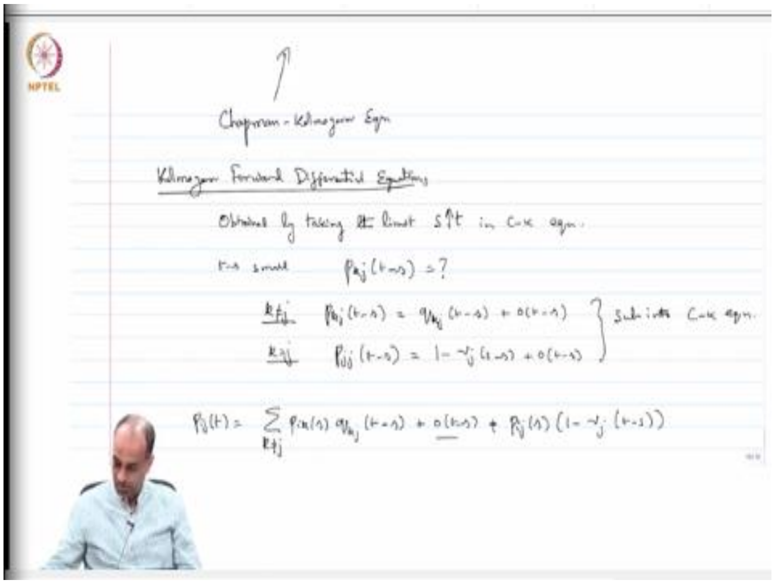

And then I have this term sum over k = j in which case I should write P ij of s times 1 - nu j times t - s plus you will have order o t - s which I have absorbed here.

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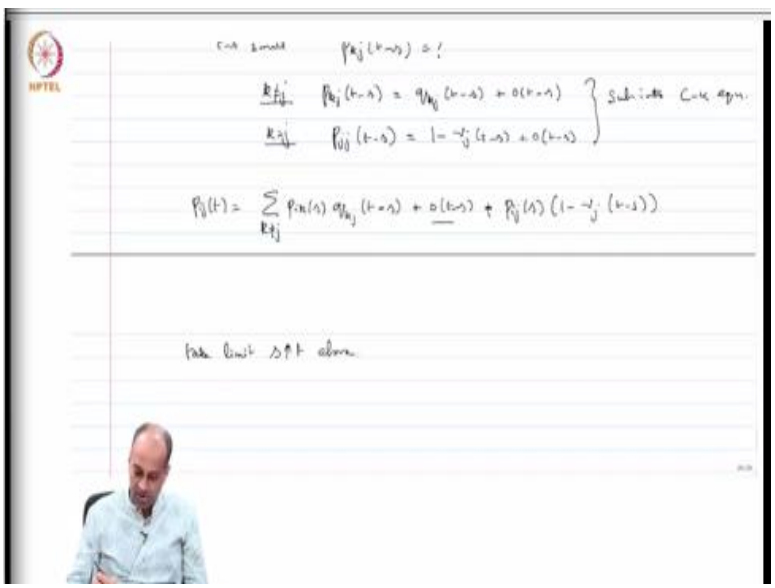

So, now I can send take limit above. So, I am going to before I take limit I should just write this as maybe I should do this a later.

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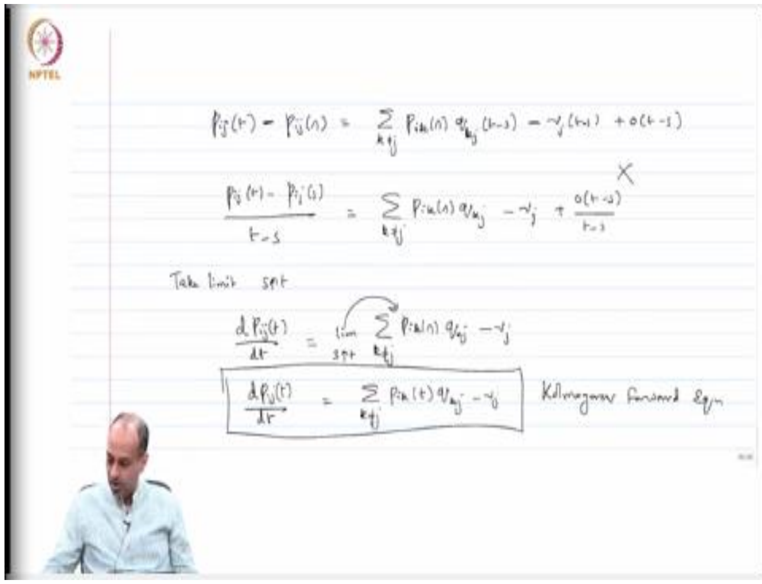

So, I should do P ij of t - P ij of s = sum over k not equal to j P i k of s q kj of t - s - nu j times t - s + o t - s. So, now you divide the whole thing by t - s in which case you will get and then take limit. So, if you then you will have P ij of t - P ij of s over t - s = sum over k not equal to j P ik of s q kj - nu j + o t - s over t - s. Now take limit s approaching t, so that if you take in that the case the limit of the left hand side will become d P ij t dt.

And that will be equal to and this term will just drop off if you take limit t tend into s, this you will get limit s approaches t sum over k not equals j P ik of s q kj - nu j. Now the question is can I send the limit into the sum? If the summation is finite in the finite state Markov chain I can certainly send it. In an infinite chain you have to be a more careful because if you are pushing a limit inside an infinite sum then you need some kind of a dominated convergence or some such thing.

Nevertheless you can validly do it as long as the nu j’s the transition rates are all bounded above then you can send the limit inside, this you can show. And in any case for a finite chain you can definitely send it inside. So, we can pretend that this is valid, so I will just write this as sum over k not equal to j P ik of t q kj - nu j. So, for each ij I can write this differential equation.

So, these are coupled, so I want to find out P ij of t I have written down the differential equation for P ij of t in terms of other P i case and similarly I can write t d P ik of t is equal to whatever.

So, I have this coupled system of linear differential equations, these are linear differential equations which i can solve for P ij of t, so this is called the Kolmogorov forward equation.

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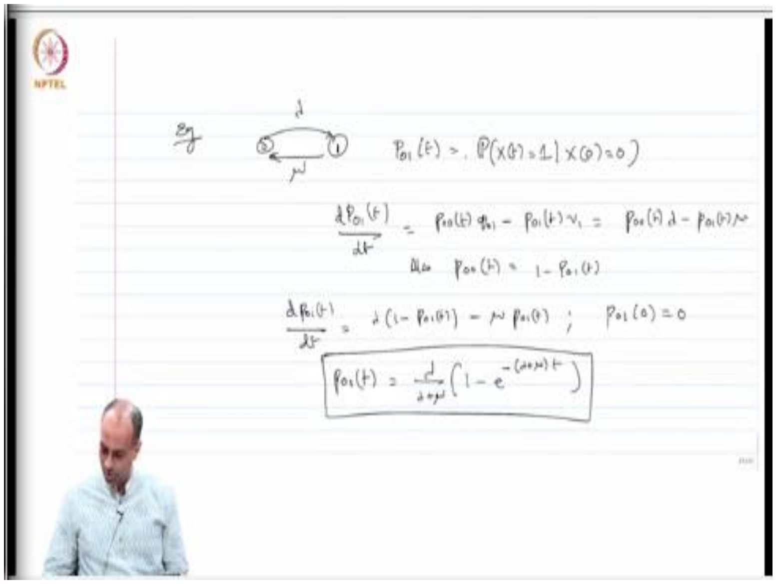

So, to just show you a simple example of how to use this? If you just take this kind of a Markov chain, let us say this is a CTMC in which we can think of this as an M/M/1 queue without any ability to store a customer. So, you either have 1 arrival like a Poisson process of rate lambda, the customer gets served and leaves there is no further possibilities for any arrival. Any arrival if you will just get dropped if the server is busy, if not the person gets served and leaves. In this case it is easy to check that basically q 01 is lambda, q 1, 0 is mu.

So, the question is can we write down P 01 of t equal to what? P 01 of t simply the probability that X of t = 1 given X of 0 = 0. So, I start the CTMC this queuing system starts empty and then I am going to look at the probability that there is 1 customer in the system at time t from finite time t. I can of course write out the Kolmogorov differential equation like so. So, I can write dP 01 of t dt = P 00 t q 01 which is lambda basically q 01 - P 01 t nu 1 that is equal to P 00 t lambda - P 01 t mu.

Now also P 00 t = 1 - P 01 t, so I can the above differential equation just becomes dP 01 t dt = lambda times 1 - P 01 t - mu times P 01 t. And I can solve this to write this is just a very simple linear differential equation. And so P 01 t and I also have the initial condition that P 01 of 0 = 0, because the Markov chain is starting at state 0 and time 0. So, I can just solve this differential equation very easily and I get lambda over lambda + mu 1 - e to the -lambda + mu t, so this is great. So, I have the probability that the q has 1 customer at time t is given by lambda over lambda + mu 1 - e power - lambda + nu t.

Of course after a very long time this goes to lambda over lambda + nu which is the steady state probability that there is 1 customer in the queue. But this is true for every time, so if I want to know what is the probability that there is 1 customer in the system at time t = 2.5? I can find out, so that is the nice thing about these Kolmogorov equations. Well, as you can imagine this is a very simple example for 2 states things become a more complicated if you have multiple states, you will have a coupled system of linear differential equations.

But in the coupled ODs are widely studied in as a branch of control or just ODs. So, there is enough literature on this, once you write down the forward differential equations it is a well understood mathematical object.

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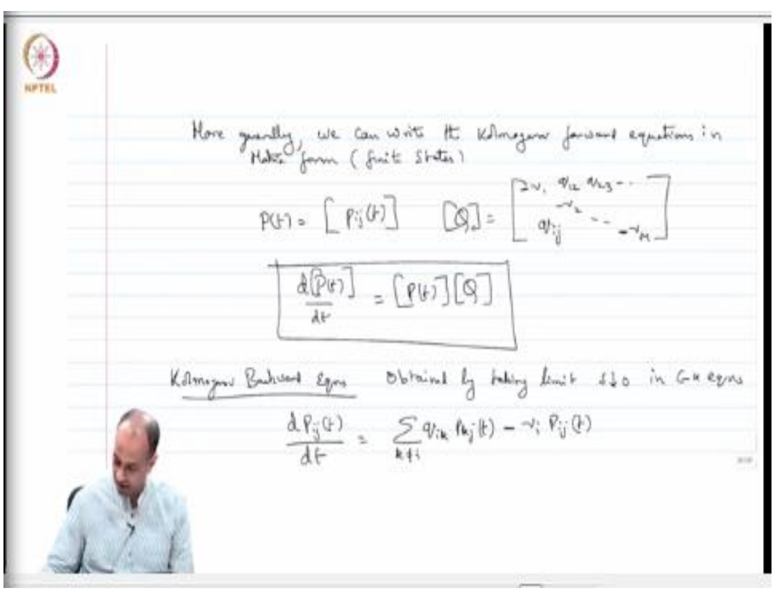

In more generally we can write the Kolmogorov forward equations in matrix form, so for finite states. So, what you do is you have the matrix of, so let us say P of t the matrix which is the matrix of P ij of t and let matrix Q be equal to the matrix of -nu 1, -nu 2 dot. dot, dot -nu let us say -nu M state Markov chain. And then everywhere else you have q 12 q 23 and so on. So, basically this guy has q ij in the ith position except that the negative the diagonal elements are all negative -nu's, -nu i's.

So, what you have is that this Q matrix has rho sums equal to 0, if you remember all the q ij is sum to nu j. And you are putting nu j at the jth diagonal position, so this is a matrix which has all the rho sums equal to 0. So, it is a non full rank matrix, it has an Eigen value 0. And this for this with this Q matrix notation you can write down the forward differential equations as dP t dt which is the matrix is equal to P of t times matrix Q. So, this is the general form of the Kolmogorov forward differential equations.

Now you can also write out the Kolmogorov backward differential equations I do not want to get spent too much time on this. So, the Kolmogorov backward differential equations, so it obtained by taking limit s down to 0 in Chapman-Kolmogorov equations and that leads to the equation dP ij dt. If you do a similar calculation sending s to 0, so you write down P ik of s as q ik times s + o s and all that. So, you get dP ij j dt = sum over k not equals i q ik P kj of t - nu i P ij of t. So, these are the Kolmogorov backward equations.

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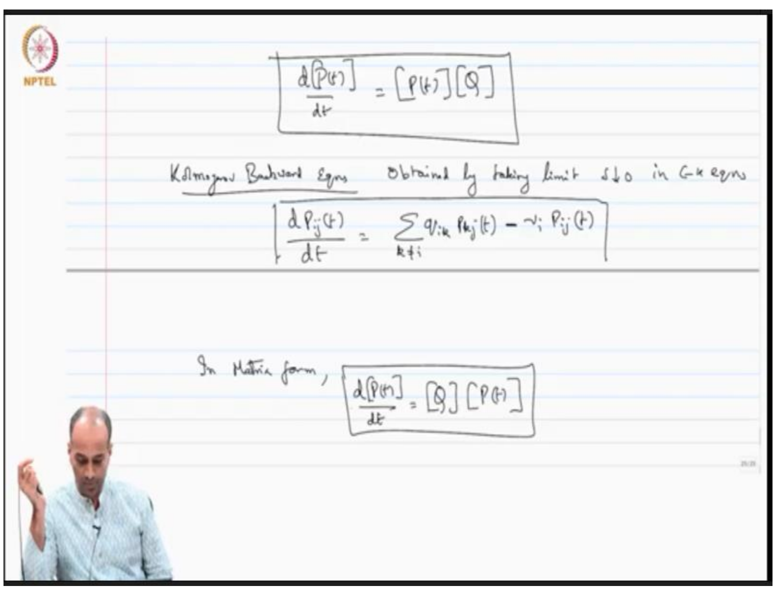

And in matrix form this becomes d P t dt = Q P of t. So, this backward Kolmogorov equation looks almost the same except the Q and P t are on the right hand side they are interchanged. But the left hand sides are actually just the same; it is just dP t by dt. So which shows that the matrix

P of t which is the matrix of the P ij of t commutes with the rate matrix Q, this Q is called the rate matrix.

So, this P t matrix and Q matrix commute, so which is actually a somewhat non trivial observation and that comes very for free from just looking at the Kolmogorov backward equations in the forward equations. Anyway this also you can solve, these are also a coupled linear differential equations. So, bottom line is that these for finite state Markov chains especially if you have small number of states you can just solve these differential equations to find the probability that you are in at any finite time t, the probability that you are in state j given that the CTMC started in state i. Like we did for the toy example of just 2 states, any finite state you can do. I will stop here.