Stochastic Modeling and the Theory of Queues

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Lecture-72
The Steady State Behaviour of CTMC-Part 4

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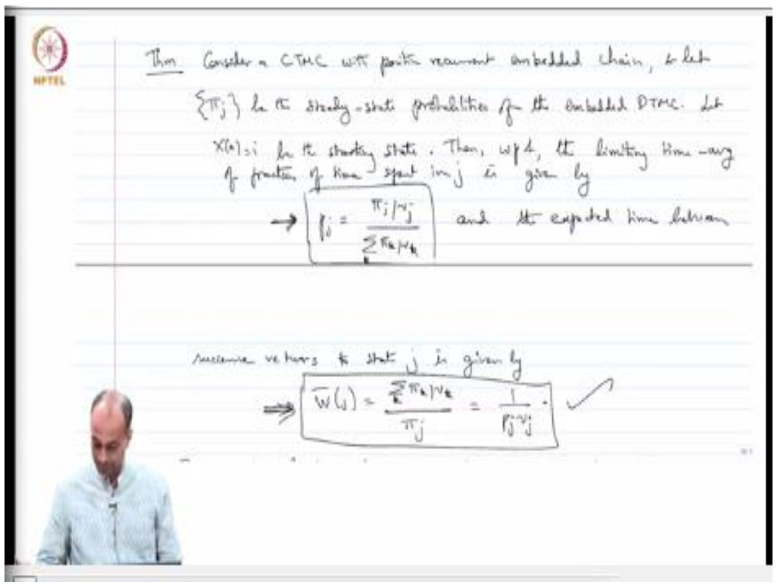

Welcome back in the last module we were discussing the steady state behaviour of CTMCs. We proved this important theorem which relates the steady state probabilities of the embedded DTMC pi j's to the steady state probability P j's of the CTMC. So, in this theorem we showed that if the embedded DTMC is positive recurrent. And if this nu case are such that sum over pi k over nu k is finite then we have a well defined set of steady state probabilities P j's that are given by this equation on the screen.

Remember that pi j has the interpretation of the fraction of transitions that go into state j. Whereas P j has the interpretation of the fraction of time spent by the CTMC in state j. We also found out that w j bar which is defined as the average amount of the expected renewal interval. The expected amount of time between 2 successive returns to state j is w bar j which is given by this equation here.

Again this assumes that sum over k pi k over nu k is finite. If this is infinite then the successive returns to j the expected time between successive returns to j is infinite. Then you will have P j = 0 then the fraction of time spent in state j will be equal to 0.

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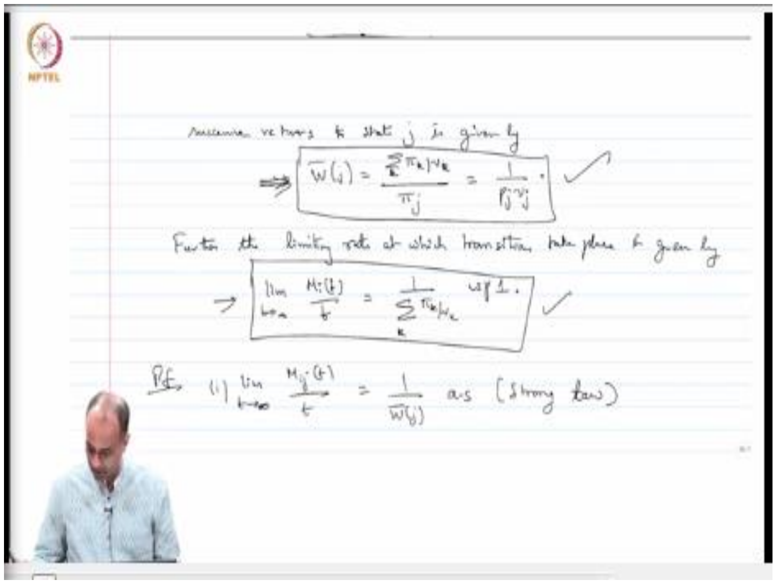

And finally the rate at which the process makes changes in states which is M i t over t, M i of t if you remember is the number of transitions made by the CTMC up till time t. So, M i t over t denotes the rate of transitions as t tends to infinity this goes to 1 over sum over k pi k over nu k. Again if this sum is finite, this is some positive number, if the sum is infinite the average rate at which the process makes these transitions goes to 0.

In that case the average rate of transition the process so sluggish that the average rate at which it goes from one state to another is 0 and you do not have any meaningful interpretation of P j, all the P j's will be 0. So, this is the result we proved last time.

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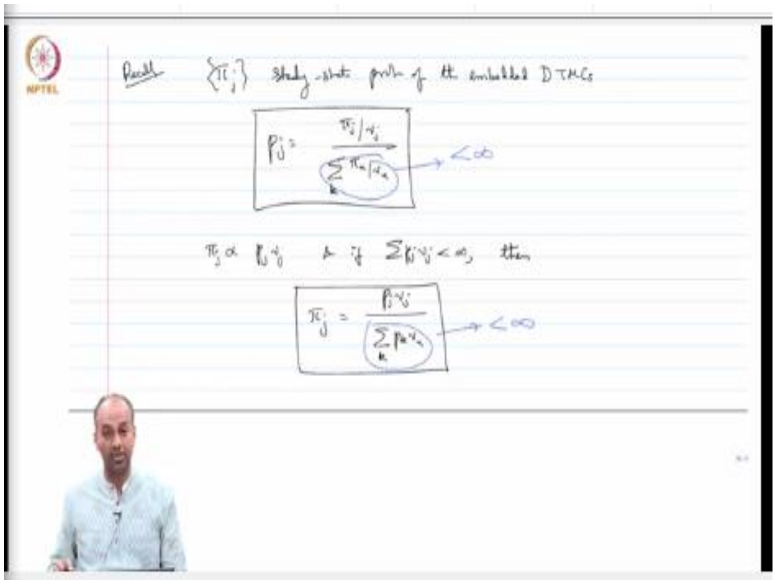

So, we can proceed now, so just to recap, so you have pi j are the steady state probabilities of the embedded DTMCs. We said that P j which is the fraction of time spent in state j or the steady state probability of state j they are both as equal to pi j over nu j over sum over k pi k over u k assuming that the denominator is finite. Then this P p j's constitute a probability distribution over the states.

So, we can also now go back, so this expresses P j's in terms of the pi's and nu's. We can also do it the other way around, so if the denominator above is finite then we can express pi j as being proportional to P j nu j except for that the normalization constant pi j is proportional to P j nu j. And if sum over P j nu j is finite, then we can write the pi j's in terms of P j's, P j = P j nu j over sum over P k and u k and all of these will be strictly positive.

So, if the sum over j pi j and nu j is finite you have the strictly positive pi j's which represents the unique steady state probabilities of the embedded DTMC. So, if you know the P j's you can find the pi j's and vice versa. If you know the pi j's you can find the P j's using this formula. Now there is just maybe I should just mention very briefly that there are 2 pathological cases where. So, we are assuming that here we have assumed that this denominator is finite and here we are assuming that that denominator is finite.

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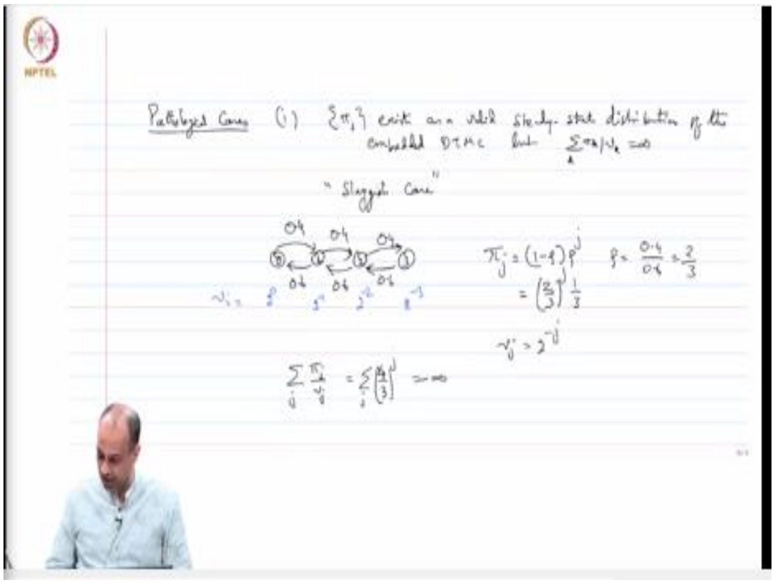

Now what if these 2 are not true? So, this corresponds to 2 extreme cases of CTMC behaviour. So, let me just briefly mention this of pathological cases. The first pathological case is when pi j exists as a valid steady state distribution of the embedded DTMC. But sum over pi k over nu k = infinity, in this case this is a case this I should write nu k. This is a case where sum of these nu k's are really small, therefore the rates at which the transition occur is very, very small in some states.

Therefore in some states the CTMC states stays for a very long time. So, this is a colloquially we can call it the sluggish case. It is sluggish because there is the embedded DTMC keeps making transitions, it is a very valid positive recurrent DTMC. But the holding times are such that some of these nu's are very small and therefore the holding times are very large. And in some of these states the process takes such a long time to move to other states that the average rate of transitions actually goes to 0.

In this case M i t over t will in fact be 0 as t tends to infinity. So, there are examples you can make up let us say you have you can have a DTMC where this is let us say I am just drawing the transition probabilities. Let us say this is my embedded Markov chain and so on, so birth death Markov chain. And the holding times are such that this is 2 power 0, 2 power -1, 2 power -2, 2 power -3 and so on.

So, the nu i's are becoming, so this is nu i in each of these, so nu 0 is 2 power 0 nu, nu 1 is 2 power -1 and so on. So, in this situation what happens to the sum over pi k over nu k? So, clearly you can easily find out that the embedded DTMC that I have drawn out is in fact positive recurrent. So, you will find that pi j is just 1 - rho times rho power j is not, where rho = 0.4 by 0.6 which is just 2 by 3.

So, that will just work out to be 2 by 3 whole power j times 1 by 3. But the nu j's are 2 power –j, so if you look at sum over pi j over nu j, this will be equal to something that is infinite. So, it will become something like 4 by 3 over j 4 by 3 power j or something like that, this is just infinite. So, what happens here is that? As the DTMC keeps a nice positive recurrent behaviour but when the DTMC goes further what happens is that?

The transitions occur so slowly in continuous time that the average rate of transitions goes to 0. And in this kind of a case, this is a sluggish case where you get all the P j's are 0. So, there is no good notion of average fraction of time spent in state j or the steady state probability of being in state j in continuous state. Of course in discrete time the Markov chain is positive recurrent.

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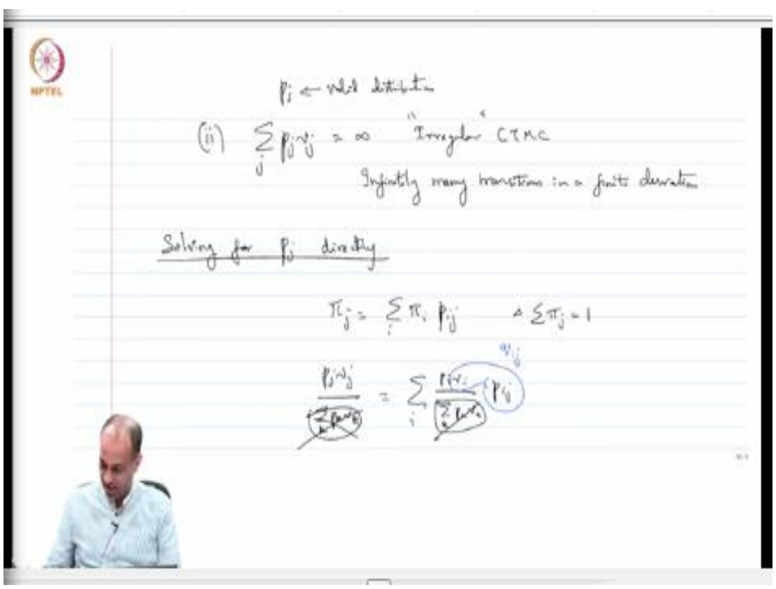

So, this is one pathological case, the other pathological case corresponds to when sum over j P j nu j is infinity. So, in this case, so let us say you are able to find P j's which satisfy the balance equations for the CTMC which I will talk about in a minute. Or you can think of P j's as the steady state probabilities of the time sample Markov chain, the delta times sample Markov chain which represents also the fraction of time spent by the CTMC in state j.

But the nu j's are such that some over j P j nu j is infinity. In this case this is called a irregular CTMC and that is because as you go through the state some of these nu's are so large and the holding times in some of these states are so small. That this kind of a process can make infinitely many transitions in a finite amount of time, so you have like an avalanche of transitions. So, in this case you cannot have although the P j's are a valid distribution P the underlying the embedded DTMC will turn out to be null recurrent or even transient.

So, there is no good notion of pi's here, so you have in this case is an irregular CTMC which goes through a infinitely many transitions in a finite duration with some positive probability. So, this is like an avalanche of transitions that happen and therefore there is no good notion of embedded DTMC steady state probabilities. Anyway, so these 2 are not very well motivated in engineering or other applications, so they are just what they are, they are really just pathological cases but mathematically they can arise.

But we will not pay too much attention to these cases, we will stick to sum of pi j over nu j being finite and sum over P j nu j being finite, so these are the nice CTMCs. Now there is 1 topic I want to address right now which is solving for P j directly. So, the one way to solve for P j is to you start with solving for the embedded DTMCs pi j and if you know nu j you can just calculate P j as pi j over nu j divided by sum over k pi k over nu k. But there is also a way to solve for P j directly in the nice case, when you do not have either the avalanche kind of a behaviour all the sluggish kind of a behaviour.

So, if you start with this let us say which is pi = pi P which is let us say pi j = sum over i pi i P ij and sum over pi j = 1. Now what you do is for pi j you substitute your pi j being P j nu j over sum over k P k u k. You get sum over i P i nu i over sum over k P k nu k P ij. So, if let us say this is finite then I can just cancel that out, so then what do I get? So, I get P j nu j = sum over i P i nu i P ij. Now what is P ij nu i? That term if you remember it is just q ij which is the rate of transitions from i to j.

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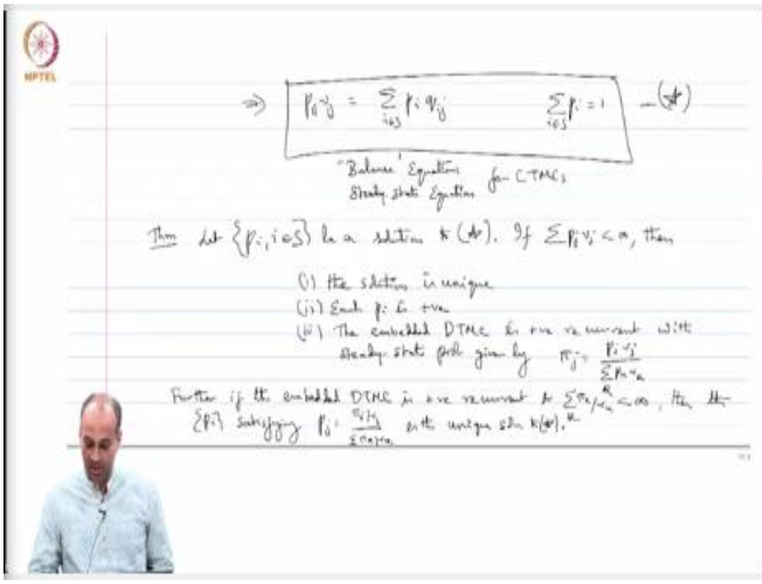

So, I can just write this as equation can be written as P j nu j = sum over i P i q ij. And of course in the nice case sum over P i = 1. So, these 2 equations can be regarded as the balance equations or steady state equations for CTMCs. So, what is this equation saying? It is saying that P p j nu j which is the probability of being in state j and making a transition. So, the rate of P j nu j is the total rate of transitions out of j in steady state.

And what is on the right is the total rate of transitions into j, P i is the steady state probability of being in i and q ij is the rate of going from i to j and you are summing over all these states. So, we are just saying that the total rate at which you get out of state j is equal to the total rate at which you get into state j; that is what this is saying. So, basically the P i's which are the steady state probabilities satisfy these equations.

And it is also not difficult to show that if there is a P j, so if you manage to solve this set of equations for some P j's which are normalized to 1. And if it so happens that sum over P j nu j is finite then you can show that P j's are in fact the unique solutions of the steady state solutions of the CTMC. This comes from the uniqueness theorems of DTMCs that we have done. This I will just state without bothering to prove it because I mean we have essentially done this in the DTMC case.

So, let this be called star, let P i let us say (()) (18:06) be a solution 2 star if sum over P j nu j is finite. Then number 1, the solution is unique that is the solution P i that you manage to find by solving the above equation star is unique each P i is positive. And 3, the embedded Markov chain is positive recurrent with steady state probabilities given by pi j = P j nu j over sum over P k nu k.

Further, if the embedded DTMC is positive recurrent and sum over pi k over nu k there is a infinity. Then the P i satisfying over sum over pi k over nu k is the unique solution 2 star. So, both sides are true, so if you manage to find a P i which is solution to the balance equations above. Then with some more P j nu j finite, then you can say that the solution is unique and each of these P i is positive and that the embedded DTMC is positive recurrent.

Now conversely if you know that the embedded DTMC is positive recurrent and sum over k pi k over nu k is finite. Then P i is which are given by flipping around the equation that we already know. Means the P i’s which are given by pi j over nu j over sum over pi k over nu k is the unique solution to the balance equations, both directions are true. And this you can prove by simply using the uniqueness theorem for DTMCs. So, that finishes our discussion about steady state behaviour of CTMCs.