Stochastic Modeling and the Theory of Queues
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Lecture-70
The Steady State Behaviour of CTMC-Part 2

So, now we can move forward as I said we will use renewal reward techniques which we are somewhat familiar with, so I will not spend too much time on this, so let me develop some notation.

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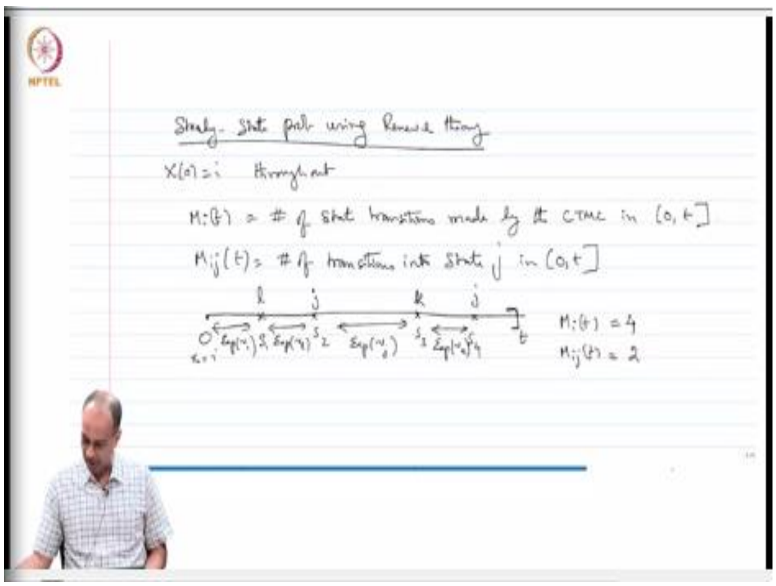

So, we will derive, so what we want to do is to derive steady state probabilities using renewal theory, just like we did it for DTMCs, this is one approach, there are other approaches as well. So, I am going to denote let us say I am going to keep X 0 = I throughout, so at time 0 the CTMC starts at state i and I am going to say let M i of t be the number of state transitions made by the CTMC in the interval 0, t.

And I am going to call M ij of t as the number of transitions into state j in the interval 0 to t, starting in state i. So, to give you a picture, so this is time 0 and that is time t. So, you starting at time 0 X 0 = i, let us say the first transition U 1 happens here, so this is your S 1. And then there is another you get into some state and then there will be an exponential corresponding to that state then there will be some other state transition S 2, S 3 and so on.

So, you look at up to 1 including t what are the number of transitions that the CTMC has made which is also the number of transitions the underlying DTMC has made, in this case is 4. So, in this case M i of t will be 1, 2, 3, 4 and if let us say that here I entered state j and here I entered state j, here I entered some state l, here I just entered some state k. So, this guy will be exponential nu l, this guy will be exponential nu j, that will be exponential nu i, that will be exponential nu k and so on.

Now M ij of t will be the total number of times I have entered state j until time t, which in this case would be 2. So, these 2 these are you can view M ij of t and M i of t has some counting processes, it is counting something. What you will see is that this M ij of t is actually a delayed renewal process, M i of t is not a delayed renewal process, it is not a renewable process at all. Because it enters different states and stays there for a different time. But M ij of t turns out to be a delayed renewal process, so we have to see that.

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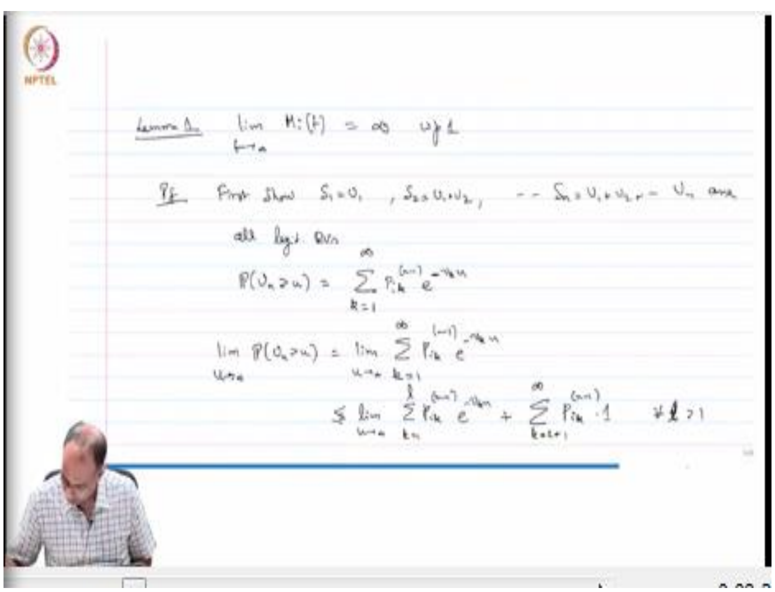

First of all you can prove a few lemmas, which I will not get into great detail you can show that lemma 1, you can show that limit t tending to infinity. If you look at this limit t tending to infinity of M i of t, this will be infinity with probability 1. The way to see that is so the proof sketch first show S 1 = U 1 S 2 = U 1 + U 2 dot, dot, dot S n = U 1 + U 2 + dot, dot, dot U n are all legitimate random variables, meaning that they are all finite with probability 1.

So, for example you can to show that if you look at this guy probability sum U n is greater than U, this is simply what is the probability that U n is greater than U? So, you have to, so U n will be an exponential with parameter sum nu k. So, the probability of that will be like this, so you have to go from i to k in n - 1 step. For underlying Markov chain should reach state k and then you have an exponential, rate of rate nu k.

So, you will have e to the -nu k u, where k maybe I should say is equal to 1 to l limit, l tending to infinity, is that right? I am just summing over all possible states and taking limit l tends to infinity. Maybe I should just write this, is just an infinite summation k = 1 to infinity. So, if you send maybe I should write this as just this is easier. Now you should look at limit u tends to infinity of probability U n greater than u, you will get limit u tends to infinity sum over k = 1 to infinity P ik n - 1 e to the -nu k u.

Now I can sum, maybe I should do this, I can sum to sum finite limit u tends to infinity sum to k = 1 to l P ik n - 1 e to the -nu k u plus the rest of the sum from l + 1 to infinity I can have P ik n – 1. And if I make, see and then you have e to the -nu ku but that guy is less than or equal to 1. So, if I make that 1, I can make this less than or equal to, it is true for all l. Now if you look at it this term.

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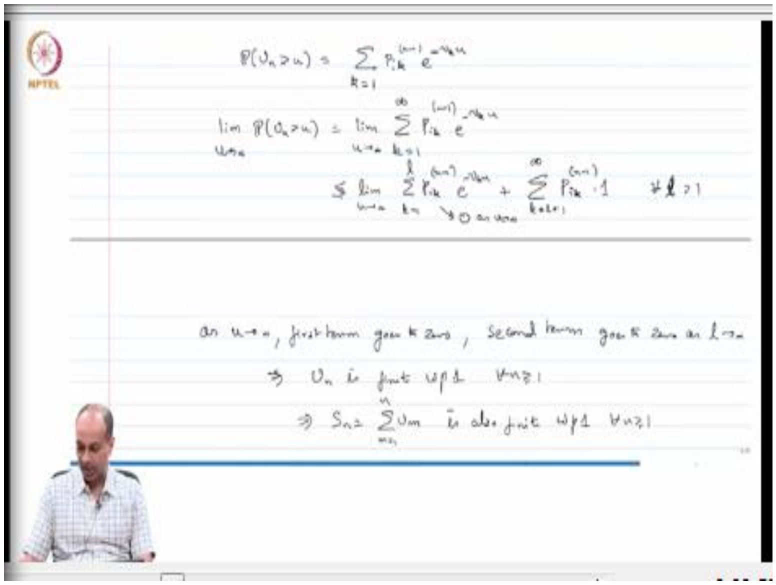

So, I have these 2 terms to handle, the first term is a finite sum and as u tends to infinity the first term goes to 0. So, this guy goes to 0 as u tends to infinity and second term is as goes to 0 as l becomes large. That is because it is a tail sum of a finite all this P i case n - 1 sum to 1 after all. So, the second sum goes to 0, if you take a large enough l. So, bottom line is that, so if you just do this carefully you can just conclude that U n is finite with probability 1.

That implies for all n, for all n greater than or equal to 1, implies S n which is simply sum over U m is also finite with probability 1 for all n. S n is the epoch of the nth transition which is we have just shown is a legitimate random variable, so that is step 1. So, the first step is involves showing that all these U i's and therefore the S n is finite with probability 1, they are legitimate random variables.

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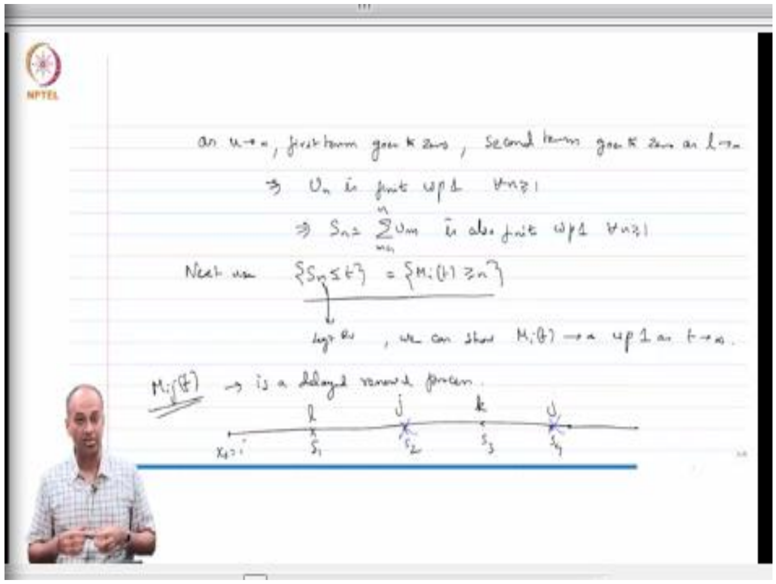

Next use the following equivalence, we know that the event S n less than or equal to t is equal to the event that M i of t is greater than or equal to n, for every sample path this is true. Now, so this is exactly now playing out like a corresponding result we proved for renewal processes, except this M i of t is not a renewal process but nevertheless this equivalence holds and using this and the fact that this S n's are a legitimate random variable which is finite with probability 1.

Then we can show that M i of t goes to infinity with probability 1 as t goes to infinity. This I think if you will go back and look at your lecture on the renewal processes, we use some continuity of probability sort of arguments and use the equivalence between S n and n of t in the renewal process case. To prove that n of t the number of renewals goes to infinity as t tends to infinity.

A similar way almost an identical argument can be used here to show that M i of t goes to infinity as t goes to infinity. Next, so this number of transitions is going to infinity, the next thing you can show is that if you look at M ij of t which is the number of start at X 0 = i and then you transition into various states. So, let us say you get into state j, so what did I draw? I think I drew I am just redrawing the picture I had before. So, i am going to get into some different states l j, k j whatever.

So, I am here in state l, here in state j, state k, state j and so on. Now the issue is what I am going to show? I am going to show that these instances of entries to j constitute a delayed renewal process. So, you are going from i to j first and then j to j, these constitute a delayed renewal process. This needs a proof because see if you may remember that in the DTMC case we said that successive entries to state j constituted in your delayed renewal process starting at state i. And we use the strong Markov property to prove that successive entries to state j constitute renewal instances.

Now a similar argument can be used here in fact in the embedded DTMC the successive entries to state j constitute renewal instances that we already know. And then the holding times are all independent across these successive intervals and dependent only on the current state. Using these 2 we can show that M i, M ij of t is indeed a delayed renewal process.

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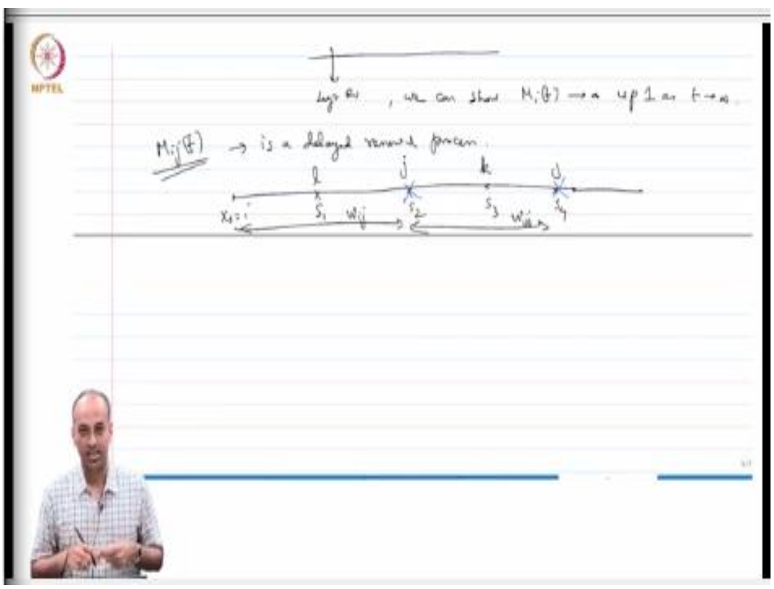

So, this involves some kind of a proof in showing that the first entry times from i to j let us if you call that interval as some t ij. And these intervals as maybe i should call it w ij and w jj. These have to be shown to be finite in the first place.

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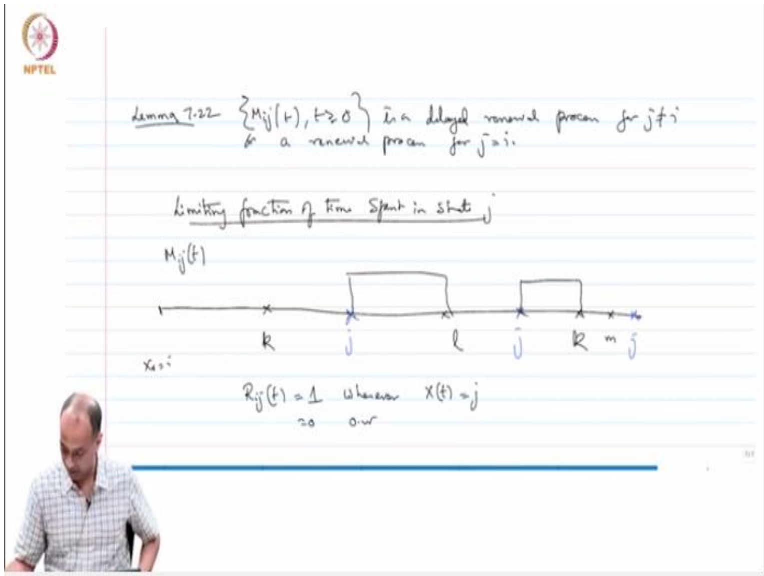

So, that also includes I mean involves a bit of an argument like I have done before, this is done in this lemma.

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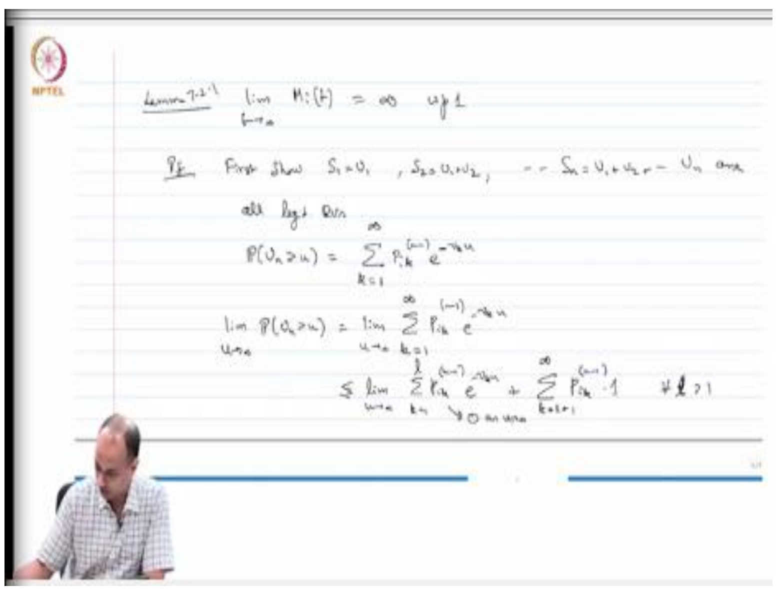

So, this is in your book, this is lemma maybe I should call it 7.2.1 in your book and this is lemma 7.2.2 in your book. M ij of t, t greater than or equal to 0 is a delayed renewal process j not equal to i and an ordinary renewal process for j = i. So, this proof I mean it requires a proof you can look at the book for arguing that all these intervals w ij and w jj are finite with probability 1. And that these w jj's, the consecutive w jj's are in fact identically distributed and independent.

This comes from the strong Markov property of the underlying DTMC and the fact that the holding times are independent across these successive entries and also identically distributed, because the distribution only depends on the current state. So, this I will not spend class time in proving. So, now I am going to look at the limiting. Now that I have a delayed renewal process, I can use some renewal reward arguments and look at the limiting fraction of time spent in state j. So, I am going to look at the delayed renewal process M ij of t. So, here X 0 = i and these are the instances where the Markov chain CTMC changes states.

And let us say these are the places where the state j is entered. And these are all some other states k, l and so on, does not matter. So, my renewal instances are whenever I enter j. Now I am going to create a reward R ij of t, whenever X of t = j and 0 otherwise, this is my reward process. So, I am in state j, so whenever I am in state j, I have a reward of 1. We had another state here m, whatever. So, the process collects reward at rate 1 whenever the process is in state j. Now I can say that the fraction of time spent in state j is simply the time average reward of this R ij of t.

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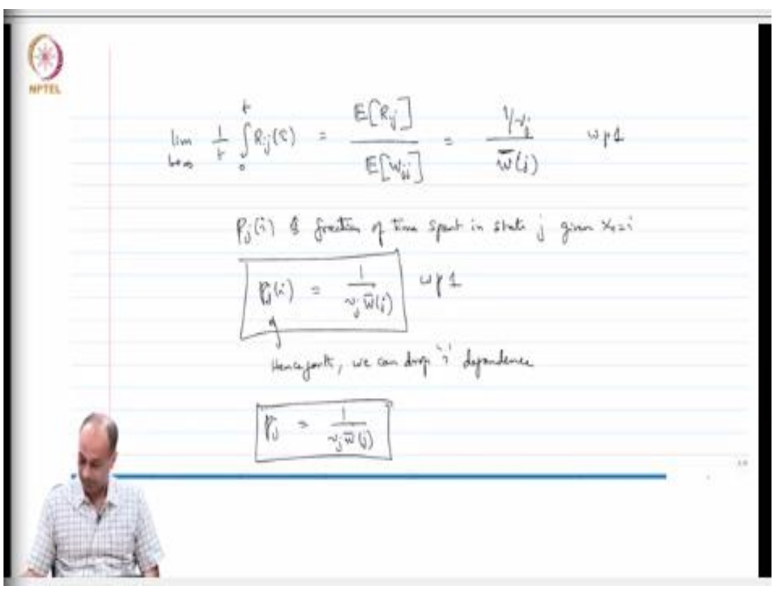

Now, so i know this, limit if I look at this guy R ij of tau limit t tends to infinity, this I know to be expected R ij which is the expected reward in 1 renewal interval divided by the width of the renewal interval. I am going to call that expected w jj or let me just call it, so as a matter of notation I am just going to call this w j bar, w j bar is the expected w jj which is the expected renewal duration.

So, that a duration like that, expectation of such a duration, this is true with probability 1. And what is expected R ij in 1 renewal interval what is the expected time spent in state j? So, that is remember that this interval is exponential with parameter nu j. So, the expected reward is simply 1 over nu j because that is the expected value of that exponential. So, there LHS is the average fraction of time spent by the CTMC in state j starting in state i. So, I am going to call P j of i which is the fraction of time spent in state j given X 0 = i, given that you started in i.

So, P j of i is your left hand side is equal to 1 over nu j w bar j with probability 1. So, what we see here is P j of i is just it does not depend on i at all. So, I can just call this P j of i as P j, we can drop i dependence. And then say that P j which is the average fraction of time spent in state j is simply 1 over nu j w bar of j. So, w bar of j is the expected renewal interval and 1 over nu j is the expected amount of time spent in state j whenever you enter state j in 1 renewal interval. So, P j is this, it will almost here excellent.

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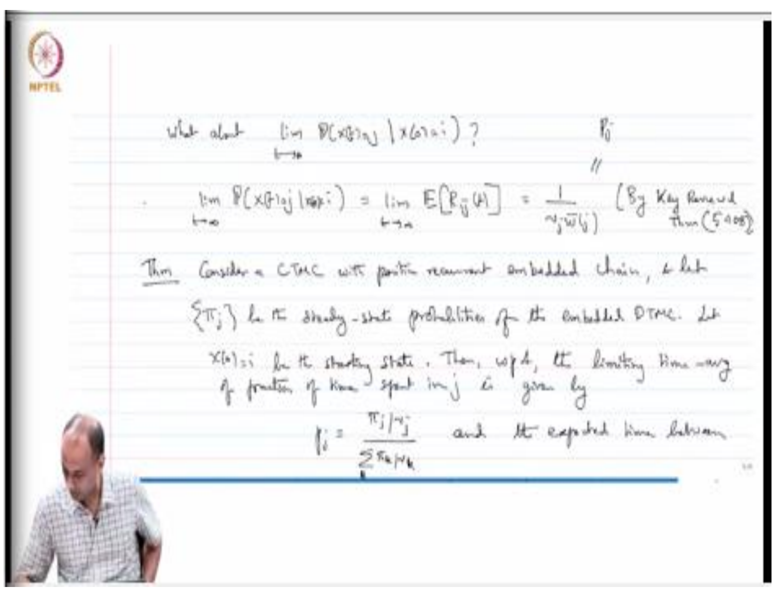

So, this is a very nice result. Now what about, so this is the time fraction, limit now what about the ensemble average? Probability of X t = j given X of 0 = i, so we have limit t tending's to infinity probability of X t = j given X 0 = i, is simply limit t tending to infinity expected R ij of t. See R ij of t is an indicator after all; it is just the expectation of the indicator that the process is in state j. And this again is equal to by k renewal theorem, it is simple, the answer will be the same, the time average and the ensemble average will be the same.

It will simply be mu j w bar of j by key renewal theorem; in your book I think it is 5108. So, what does this mean? So, the probability that I find myself in state j after a very long time having started in any state i is independent of state i and is equal to 1 over nu j w bar j, so this is a very nice result. Now this w bar j is not we have to characterize it in terms of something that we understand about the process.

So, what have we established so far? So far we have established that the fraction of time spent in state j exist almost surely and is equal to 1 over nu j w j bar and this P j is also the steady state probability of being in state j after a very long time. So, maybe I should say, so this is equal to P j this guy is equal to P j which is above. Now but I want to find out this w j bar in more that I understand in terms of possibly the nu nu i's and P ij's or whatever.

I want to characterize it in terms of the known parameters of the process. So, that is done as follows, this is an important theorem. So, consider CTMC with positive recurrent embedded chain and let pi j be the steady state probabilities of the embedded Markov chain. Let X 0 = i be the starting state, then with probability 1 the limiting time average of fraction of time spent in j is given by, I already stated the answer some time ago, so P j = pi j over nu j over sum over pi k over nu k.

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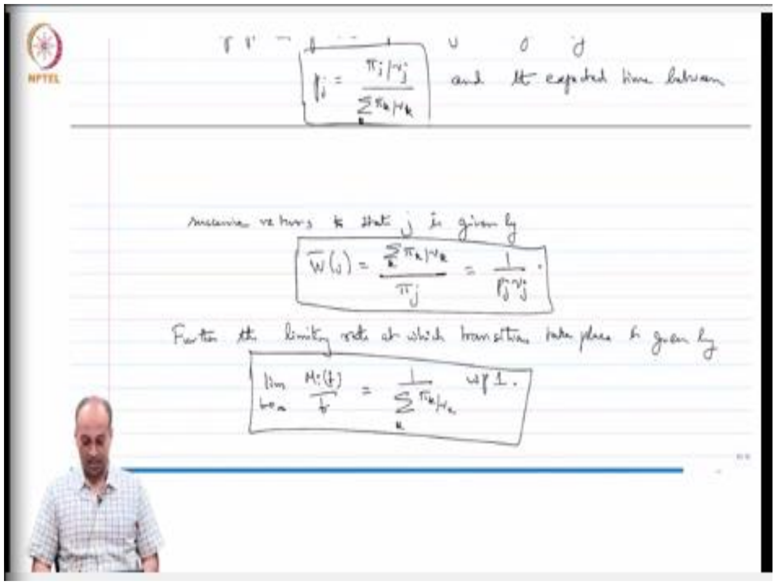

And the expected time between successive returns to state j is given by w j bar = sum over k pi k over nu k divided by pi j which is equal to 1 over P j mu j. Further the limiting rate at which transitions take place is given by limit t tending to infinity M i of t over t = 1 over sum over k pi k over nu k with probability 1. So, we are finding a number of answers here, of course the steady state probability this we already encountered.

I mean we have to prove it but this answer we already knew, this is expected w jj or w j bar and this is the rate at which the process makes transitions. There is a for every CTMC there is an average rate at which the transitions take place and that is given by 1 over sum over k pi k over nu k.