**Stochastic Modeling and the Theory of Queues**
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**Lecture-67**
# Introduction to Continuous Time Markov Chains

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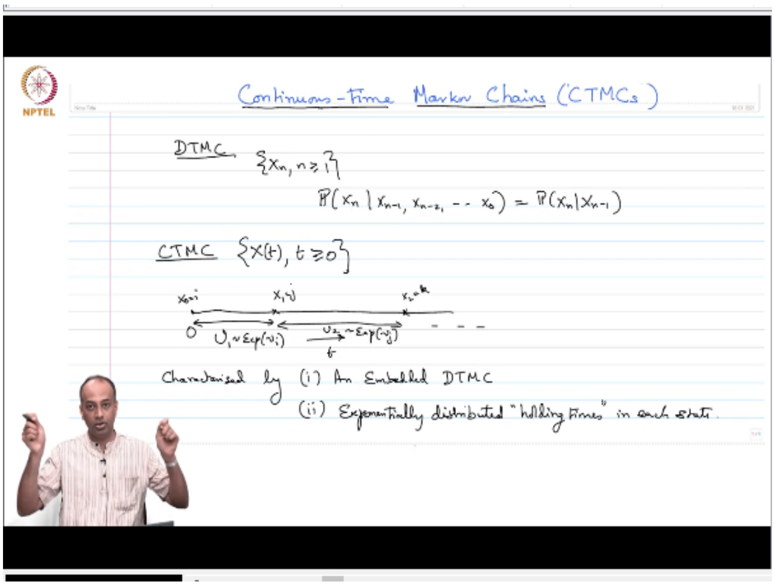


Welcome back. From today we will start discussing continuous time markov chains, otherwise known as continuous time markov processes. So, what is a continuous time markov chain? So, far we have studied DTMC's discrete time markov chains. If you remember so for a discrete time markov chain X n, so the time index is n for a discrete time markov chain and as we know that it satisfies probability X n given X n – 1, X n - 2 etcetera is just probability of X n given X n – 1.

For any values of the states in the past only the previous state matters. So, this is what a DTMC is. A CTMC or a continuous time markov chain or a continuous time markov process as it is sometimes called is something similar so it works in continuous time. So, I will broadly tell you what it is. So, X t its time is now continuous. Let us say t greater than or equal to 0. So, what is this characterised by?

Before I give you a formal definition; so pictorially it is easy to tell you what it is. So, that is time 0 and this is continuous time. A CTMC is characterized by something known as an embedded markov chain. So, it is characterized by 2 things an embedded DTMC and 2 exponentially distributed holding times in each state. So, CTMC is characterized by an embedded DTMC and an embedded markov chain and exponentially distributed holding times in each state.

So, there is an underlying markov chain called the embedded markov chain which is evolving you can think of a discrete time in which the embedded markov chain is evolving, every time the markov chain enters a particular state, the embedded markov chain enters a particular state. The continuous time markov chain stays in that state for an exponential amount of time. And then after this exponential amount of time the underlying DTMC goes to a different state and so does the CTMC and again CTMC holds for an exponential amount of time depending on the state that the embedded DTMC goes to.

So, let me just draw this out. Let us say that I start with X 0 = i then there will be a exponential holding time. Let me call this U 1. This will be exponentially distributed with a parameter nu i which depends on the state X 0 = i conditioned on X 0 = i, the holding time is some exponential random variable which is parameterized by some nu i, some positive number nu i.

Then after this exponential amount of time the underlying DTMC goes to some other state, let us say X 1 = j. This state that this embedded DTMC transitions 2 is independent of the exponential random variable that just happened; it is independent of U 1. So, the DTMC transitions to let us say some state j with some probability P ij, the P ij’s are the transition probabilities of the embedded DTMC. Again you have another holding time, let us call this U 2 and this U 2 will be exponential with parameter nu j which the parameter nu j depends on the state X 1 = j.

And again the underlying markov chain will go to some state k with probability P kj, this exponential nu j is independent of the next state and so on. So this is how the CTMC proceeds. So, you have exponential holding times in each state with parameter dependent on what state the embedded markov chain is in and after that exponential is finished both the embedded markov chain and the CTMC move to some other state j with probability P ij.

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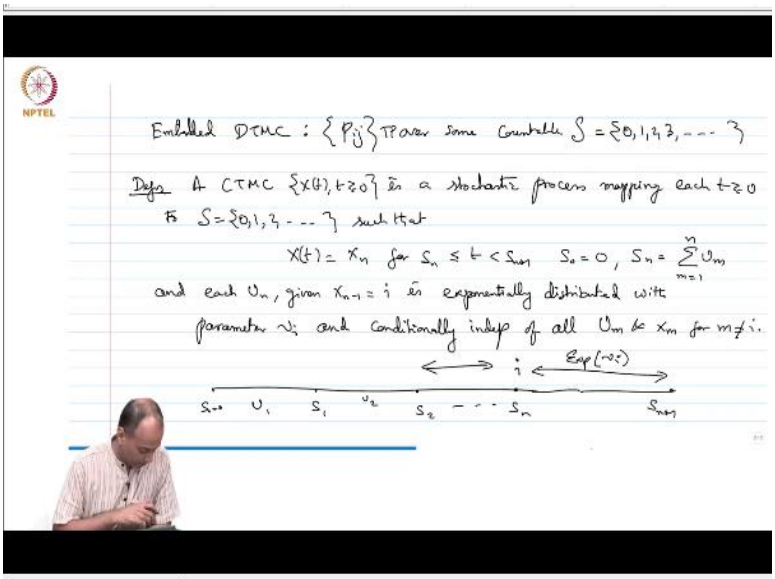


So, this embedded markov chain so let us say is P ij is the transition probability over some countable state space s which we can by default take it as 0, 1, 2, 3 dot, dot, dot. So, we are looking at the embedded markov chain evolving on some state space which is we are taking it as 0, 1, 2, 3 dot, dot, dot and P ij’s are the transition probabilities. These are the transition probabilities over some quantum state space.

Now I want to define this CTMC, CTMC X t the stochastic process mapping each non negative time to I can say s which is let us say 0, 1, 2 dot, dot, dot, such that X of t = X n for s n less than or equal to t less than s n + 1 s 0 equals 0 and s n I have to tell you what s n is. S n is the sum of these exponentials m = 1 to n and each U n given let us say X n – 1 = i is exponentially distributed parameter some nu i. Nu i depends on i obviously and maybe is a conditionally independent of all U n and X n for m not equals i.

So, this is the definition, so let us revisit this. A CTMC is a stochastic process X t for each X t takes one of the values X n, where X n is the state of the embedded markov chain. So, just to draw this out in picture again. So, you have s 0 – 0, this is your after a certain U 1 amount of time, let us say this is s 1 and then after U 2 amount of time you have s 2 and so on. Now if you just look at this so let us say this is s n let us say that is s n + 1.

So, each s n is equal to the sum of the first n u i's and each U i conditioned on the state X n - 1 being equal to i. Let us say you are in some state i here, with the markov chain is in some state i here, the underlying markov chain embedded markov chain is in state i and that is your s n + 1 that is exponential with parameter nu i. And this exponential random variable is independent of the other U's and it is also independent of condition on X n = i if independent of the other X i's. That is what a CTMC is. Now we for the embedded DTMC I mentioned that it is evolve on the state space s.

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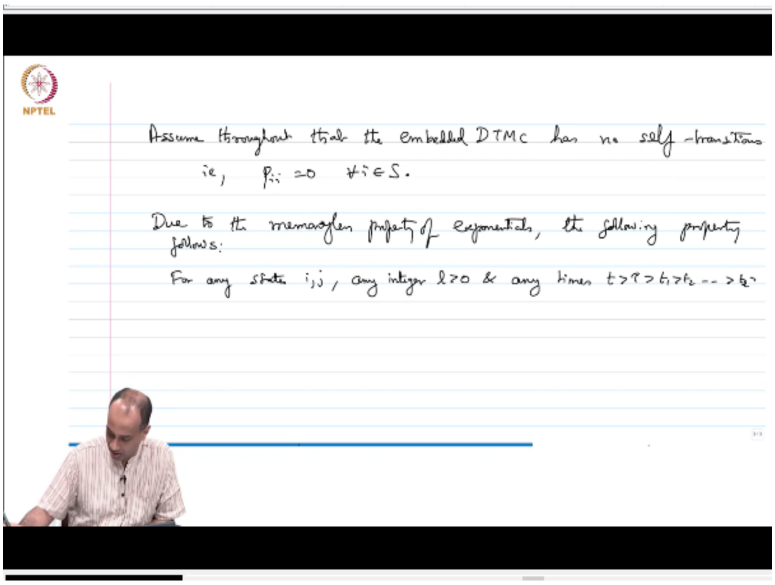


We will assume that embedded DTMC has no self transitions that is pii = 0 for all i, we are making this assumption for the embedded markov chain. That is because if the embedded markov chain transition from i to itself in the continuous time process the X t remains in i and you do not even see a transition really. So, if you really want to see a transition in the continuous time process you want to enforce that the embedded DTMC actually goes to a different state.

Now this is not a very restrictive assumption and you can actually prove that if you do have a self transition also you can just reduce it to a CTMC which does not have self transitions. By just redefining the nu is appropriately. So, this is not a very restrictive assumption. So, the nice thing about this is that if you have self transitions you do not see it in the continuous time process you cannot see it as a transition because the process continues to be in the state i and the other is that there is a nice equivalence. If you do not have say this self transitions there is a nice equivalence between specifying the exponentials U i and the P ij’s.

And the sample parts of an embedded markov chain xn and the U i’s and the sample path of the X t process. There is a one to one relationship between the X t sample path of the CTMC and the X n sample path of the embedded markov chain and the U i’s. The X n’s and the U i’s together will specify X t and vice versa. That equivalence also follows if you make this assumption. This is not a restrictive assumption again.

As I said if you do have P ii non zero you can tweak the nu i's to make P ii 0 and the statistical properties of the process will remain the same. So, this is an assumption we will make going forward that the embedded DTMC does not have any self transitions. So, what can be shown is that due to the memory less property of exponential is the following property follows. You can easily show this, for any states i, j, any integer l and any times t greater than tau greater than t 1 greater than t 2 dot, dot, dot, greater than t l.

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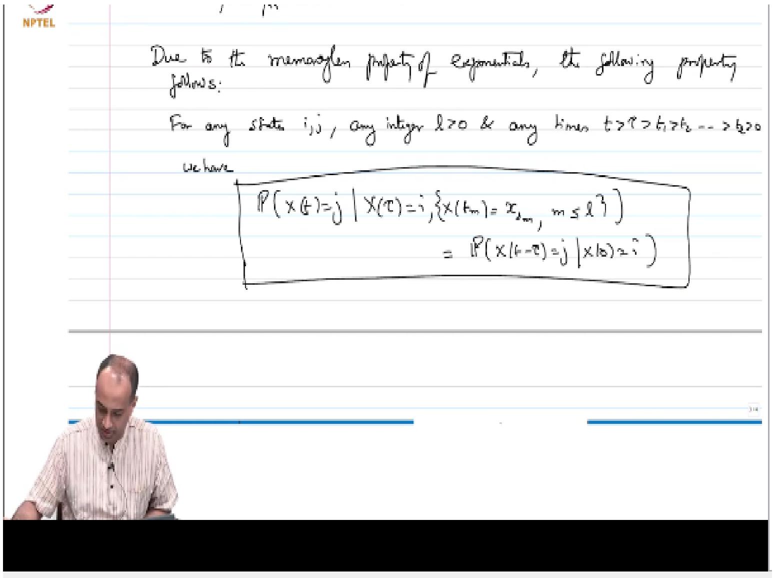


We will have probability X of t = j given X of tau = i, X of t m is equal to some X s m for m less than or equal to l. This will simply be equal to P X t - tau = j given X of 0 = i. So, what do we have here? So, this is a markov property in continuous time that is why this is called a continuous time markov process.

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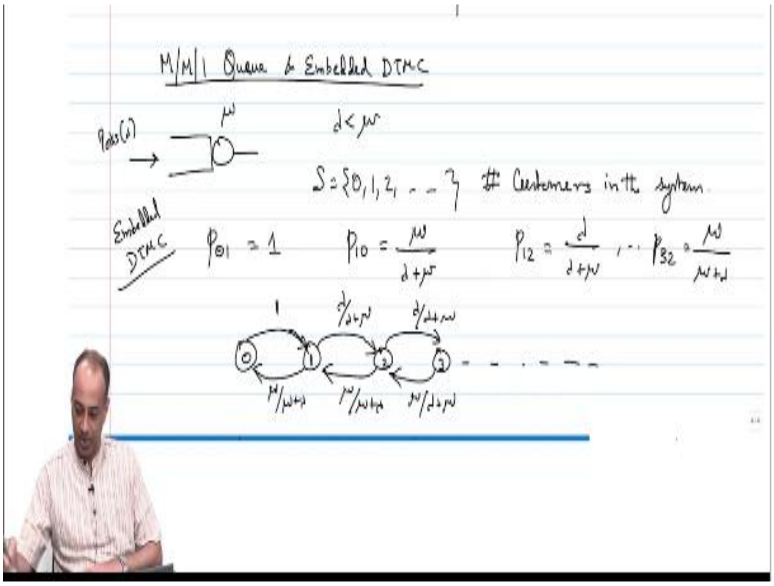


So, what this property is saying this you can easily show so that is time 0. I am looking at some times what are these, so I am looking at t l a whole bunch of times. So, let us say that is t 2 that is t 1, that is tau and that is t. These are all various points in time. I am looking at the probability that X t equals, so this is X t that is X tau and then you have a whole bunch of the process is in various states X of t 2 dot, dot, dot.

We are looking at the probability that X t = j, given X tau = i and a whole bunch of values for these states. What we are saying is that only the most reason conditioning matters that is number 1. So, all the previous conditioning these guys can be thrown out, that is what we are saying it. So, conditioned on X tau all the previous values are irrelevant and that is not all, it is as though this is again because of the first property is the markov property.

The second is that it is as though time just begins at tau, it is as though you are starting the process X 0 = i and you are looking at probability of X t - tau being equal to j, it is as though the process just started the zero time of the process is at tau and this is true for any times t tau t 1, t 2 etcetera and for any 2 states i and j. And this property follows from the property underlying markov property of the embedded markov chain and the memory less property of the exponential random variables.

This you can easily show. Next I want to show a simple example let us say, so M/M/1 queue and the embedded markov chain. As you know an M/M/1 queue has a server with exponential mu and a poisson arrival process at rate lambda and the arrival process is independent of service times. We are assuming of course that lambda is less than mu, so that the arrival rate is less than the service rate, otherwise the queue will be overwhelmed.

In this setting actually an M/M/1 queue is actually a continuous time markov process. And the embedded markov chain is just the number of customers in the system. So, the embedded markov chain can be thought of as the transitions from let say X n - 1 to X n in the embedded markov chain happens either upon an arrival or a departure. So, here the state space is again 0, 1, 2 etcetera which is the number of customers in the system and let us look at what kind of transitions are possible?

Let us look at P 01, this is the for the embedded markov chain transitions. So, this is for the embedded DTMC. P 01, see if you are in state 0 the only state you can go to next is 1, because from state 0 remember there are no self transitions in the embedded markov chain. So, the queue is in state 0 the only non trivial transition it can go to is 1. So, P 0 1 is 1, this is the embedded markov chain remembers, but if you look at P 10 for example, so there is already one customer in the system, what is the probability that the next transition the embedded markov chain which corresponds to the evolution of the number of customers in the system?

So, you are looking at the probability that the system goes from having one customer to having no customers. This happens if the customer in service completes this service before the arrival of the next customer. Remember that the customer in service this one person who is in service has an exponential service time of rate mu and the incoming customers are up independent poison process of rate lambda.

So, there is a competition between the service exponential which is mu exponential and this arrival exponential which is a lambda exponential. Of course these exponentials are independent. So, the probability that the service exponential wins is nothing but mu over mu + lambda. This we already know from our first initial study of poisson processes, independent exponentials, racing exponentials if you know so and so on.

Now if you look at what is P 12, you are looking at now there is 1 person in service instead of that person leaving first you are looking at the probability that in fact there is a new arrival before the service of the first customer completes. This is obviously lambda by lambda + mu, it is simply a 1 minus the probability that the serve is exponential wins which is the probability that the arrival exponential wins which is lambda over lambda + mu and so on.

So, now you can P 32 and all is not very different, it is the same story mu over mu + lambda and so on. So, if you look at the DTMC which is embedded DTMC you get a structure like this, this is the embedded DTMC. So, from 0 you can only go to 1, because we are not allowing self transition. Please remember that transition from 0 to 0 is not counted, so the process remains in 0 and the only next distinct transition that can happen is the arrival of a new customer.

So, this has probability 1, this is indeed discrete time and of course this probability is mu over mu + lambda, that probability is lambda over lambda + mu that is mu over mu + lambda and so on. So, this is the embedded DTMC for the M/M/1 queue and what are the nu i's. So, these are the P ij's and so what are the nu i's? If you look at this nu, so if I am in state 0 nu 0 is simply lambda.

Because if I am in state 0 I am waiting for a new customer to come in, so what is the amount of time I wait until a new customer comes in to an empty system? It is exponential with parameter lambda. So, the rate at which a transition out of state 0 is lambda. Now what is nu 1? Nu 1 corresponds to the rate at which I transition out of state 1. So, how long do I say stay in state 1 is the question?

So I stay in state 1 until either there is a new arrival in which case I go to state 2 or until the person in service completes a service. That is exponential with rate mu. Now so there are 2 competing exponentials, so the total rate at which I get out of the state or the amount of time I spend in the state is exponential with parameter lambda + mu. In fact you can argue that nu i = lambda + mu for all i greater than or equal to 1.

First state alone I am waiting for the lambda exponential to fire, waiting for an arrival. In all other states I am waiting for either an arrival or the service to complete. So, there are 2 competing exponentials, so nu i is lambda + mu. So, I have completely for the CTMC corresponding to an M/M/1 queue, this is the embedded markov chain and these are the nu i's. I hope this example is clear.

Of course you can now solve for the embedded markov chains embedded DTMC's, steady state probability you can easily determine by writing out the balance equations. In fact if you look at this DTMC you can clearly see that this is a Birth-Death markov chain, the embedded DTMC is Birth-Death and you can actually just easily write down the local balance equations and figure out what the pi's are.

In fact for the embedded DTMC you will have pi 0 = 1 - rho whole divided by 2 and pi n = 1 - rho square over 2 rho to the n - 1 for n greater than or equal to 1, where rho is just the load on the system lambda over mu, which is assumed to be less than 1. This you can easily solve. Now what is this pi n represent? So, in the discrete time markov chain of course pi n is the steady state probability of being in state n of there being n customers in the system.

But in the continuous time markov chain this pi n does not represent the probability of there being n customers in the M/M/1 queue. This pi i represent the number of transitions going into state n. This is not the fraction of time spent in state n, I want to make this very clear. See you have different news, different holding times in different states. So, this pi i is the steady state probability of the embedded markov chain which represents the fraction of transitions that take you to state n.

And because these nu i’s are different, there are different holding times in different states these may not represent the fraction of time spent in state and in the continuous time process. Note pi i obtained let us say as above by solving the balance equations for the embedded DTMC represents the fraction of transitions into state i, maybe I should just say pi i like that, state i in the CTMC.

See in the DTMC case the number of the fraction of transitions into a state is also is the fraction of time spent in that state in a DTMC. In a CTMC that is not true, the fraction of transitions that go to a state i is not the same as the fraction of time spent in the state i because some of these nu i‘s are different from the other nu i’s, nu j’s. So, you have to wait them appropriately to get the fraction of time spent in each state. So, we have not gone there yet.

I am only talking about the pi i’s, if you solve the embedded DTMC steady state probabilities you get some pi i, some probability distribution over the states. What does this represent? It represents the fraction of transitions into state i, it does not represent the fraction of time spent by the CTMC in state i. So, of course in all this I am assuming that the embedded DTMC is irreducible and positive recurrence.

So, that I can solve this pi p = pi and get some pi i's and the pi s you also obtained are the fraction of transitions in the CTMC going into state i and it is not the fraction of time spent by the CTMC in state i. And what is the fraction of time spent by the CTMC in state i; we have to work out separately, that is an important topic, we will get to later.