Stochastic Modeling and the Theory of Queues

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Lecture-64
The Reversibility Markov Chains

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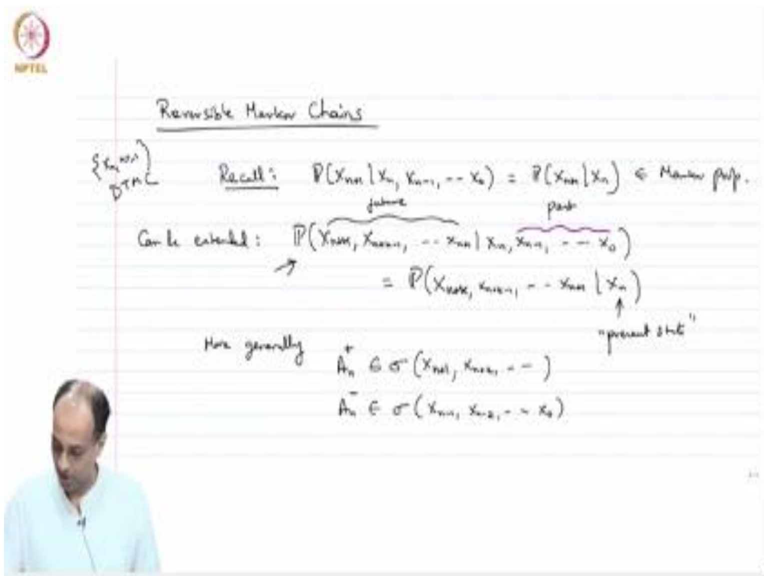

Welcome back, in this module we will discuss the very important topic of reversibility and reversible Markov chains. Intuitively speaking a Markov chain is said to be reversible if the statistical properties of a chain with time running in one direction is indistinguishable from the statistical properties of the Markov chain when time runs in reverse. And we will see when that happens in a little bit? So, that is the topic of this discussion.

Now I just want to start with recalling for you, a Markov property which says that P X n + 1 given X n, X n - 1 dot, dot, dot X 0 is simply equal to P of X n + 1 = given X m. This is no matter what values these X n's take, this is true for all n and for all values of the state X and X n - 1 etcetera. This is the Markov property, this can be easily extended. To look at this, you can look at probability of X n + k X n + k - 1 dot, dot, dot X n + 1 given X n, X n - 1 dot, dot, dot X 0.

This you can show from if X n is a DTMC. For any Markov chain you can show that this is simply you can just use conditional probabilities and the Markov property. To say that X n + this will be equal to -1 dot, dot, dot X n + 1 given X n which means that you can basically just forget about all of that, conditioning on all of that can be thrown out. This does not require any further assumptions, this is true for I am just using the Markov property and you can write it out in terms of conditioning and use the Markov property repeatedly to get this.

This is just a simple exercise; you may want to do this fully yourself. Now I want to think of this X n as the present state. And all of this guy will be all of these guys will be the past and all of this will then be the future. So, more generally, so let me denote, so at time n I want to denote by A n +, any event which is determined by things in the future. By future I mean X n + 1, X n + 2 etcetera.

So, index n X n is my present state, index n is present time and A n + refers to a time n anything that is determined by future states. To be a little more precise A n + is an event measurable under the sigma algebra generated by X n + 1, X n + 2 so on. And A n - as some event determined by the past. So, more precisely it will be some event in the sigma algebra generated by X n – 1, X n - 2 dot, dot, dot X 0. So, if you write the Markov property, so if you write that equation in this A n +, A n - sort of notation what you can show is that?

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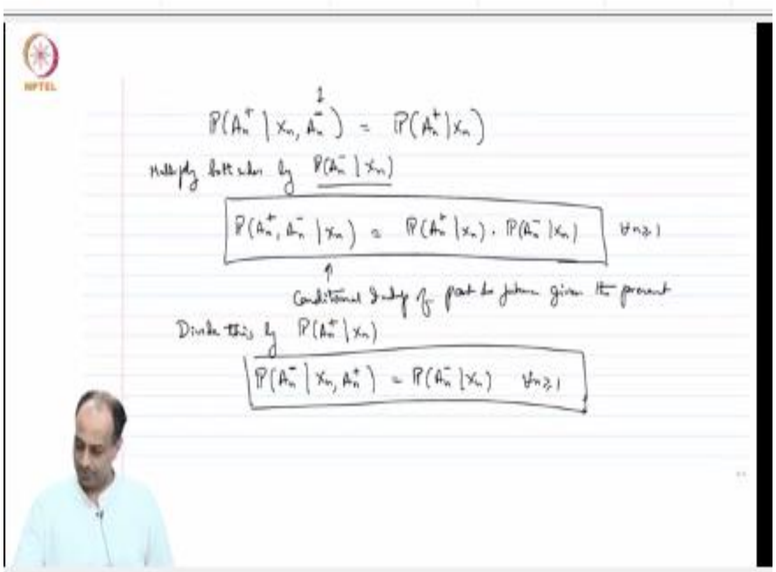

Using this Markov property, we can show that probability of A n + given X n A n - simply equal to probability of A n + given X n**.** Essentially this A n - business can be thrown out, the pass can be thrown out, it only matters where you are presently. And if you multiply this, multiply both sides by P A n - given X n and what will you get? You will get P A n +, A n - given X n I am just multiplying the left hand side of the above by that term.

You will get P A n + given X n times P A n - given X n. So, in some sense the most intuitive way to understand this is true for all n greater than or equal to 1. This is in some sense the most intuitive way to understand what a Markov chain really is. So, X n is your present state, you are in X n is in some state let us say i, it does not matter what state. A n + is things in the future, A n - is things in the past.

So, A n + can be any event measurable under sigma of X n + 1, X 1 + 2 etcetera, A n - is similarly measurable under all random variables from the past. You have the conditional independence, this is saying that there is conditional independence of past and future given the present. So, the conditional dependence of the past in the future given the present, this is what Markov property really means.

So, we knew all this, we sort of knew all this, I have just writing it down in a particular notation. Now why am I doing all this? You will see very soon, so let us go ahead and divide this by P A n + given X n then what do I get? I will get P A n - given X n, A n + = P A n - given X n. Assuming that this probabilities are non-negative, I am just dividing by P A n + given X n, so what does this mean?

This is saying that given the present and the future the probability of A n - is simply the probability of A n - given the present, the future is irrelevant. So, in some sense what this is saying is that if times are to run in reverse, in which case? So, if time runs like this, n + 1 is future and n - 1 is the past. But if time runs the opposite direction, then n + 1 becomes the past and n - 1 becomes the future at time n.

If you are at time n running this way n – 1, n + 1, so n + 1 is future but if time runs this way n + 1 is the past, then there is n present, n - 1 is the future. So, if time runs in reverse, what we are saying is that the past does not matter, past here is A n + if you think about it, in reverse time A n + is the past. So, this is some kind of a Markovian property when time is run in reverse for a Markov chain.

So, this is basically saying that if you have a Markov chain running in forward direction and you note down all it is states. And you record this as a tape and you run the tape in reverse then the reverse also has a certain Markov property. In the sense that, in the reverse time future condition on the present is independent of the past, where the future is A n - now because time is running in reverse. And the pass is A n + when time is running in reverse, so this is very nice.

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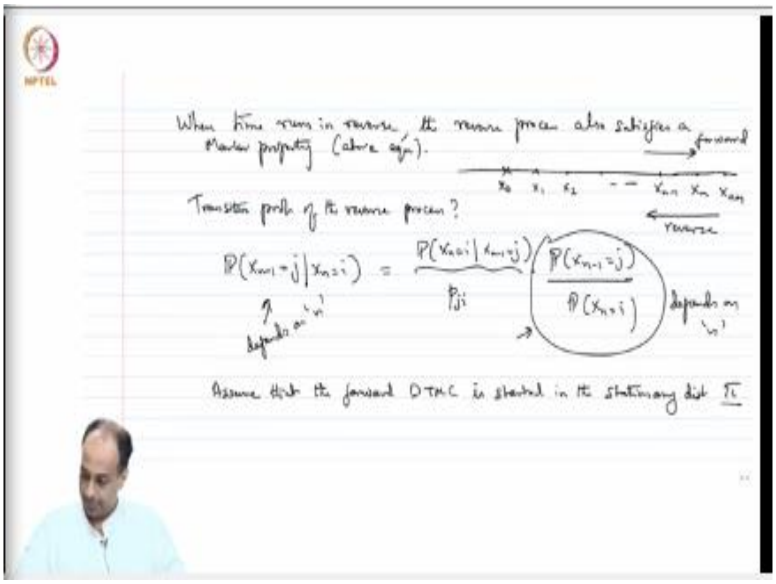

So, the reverse process, so when time runs in reverse, reverse process also satisfies the Markov property that is what the above equation is saying. Now, so if the above equation is saying that if you are, so a process is running like this, you see a bunch of states X, 13 X, 14 X, 15 X, 16 and it time in reverse you will see X 16, X 15, X 14, X 13. We are saying that even in reverse time a certain Markov property holds as given in that equation starts.

Now if the reverse process is also has a Markov property, then we have to ask what it is transition probabilities are? So, let us say we are given the transition probabilities of the forward running Markov chain which is the original X n process. Now we are, so X 13, X 14, X 15 runs according to some P ij transition probabilities. Now I run it in reverse X 16, X 15, X 14, X 13. Now these transition probabilities what are they? So, that is our next task.

So, we know that it satisfies the Markov property, we want to find out it is transition probabilities. So, what are the transition probabilities of the reverse process? We know that reverse process satisfies Markov property, then we want to find out what it is P ij's are? Let us say the forward chain has P ij as the transition probability, the reverse chain has some other transition probability, we want to know what it is transition probabilities are?

So, I want to calculate this i, so let us say so I want that. Because when you are running let us say when your Markov chain is running like this, let me just draw. So, this is let us say X 0, X 1, X 2, X n – 1, X n, X n + 1 etcetera, so this is forward time and that is reverse time. So, I want to look at the probability that X n - 1 = j given X n = i, so this is like a transition probability of the reverse process.

This I can just use Bayes' rule basic condition is conditional probability to write X n = i given X n - 1 = j over P X n - 1 = j times probability that = X n = i, I beg your pardon. I think I got this wrong, I got this wrong, I should write P X n - 1 = j over P X n = i. This is simply the definition of conditional probability. So, now, this is the transition probability of going from i to jin reverse time.

And what is that? That is the transition probability of the forward chain. So, this is simply P j to i, P ji, so I am going to assume that the forward chain is a Markov chain, the homogeneous Markov chain with transition probability P ji, so this is just the probability of going from j to i. Now the issue is that this term depends on n, correct. Because the probability of being in state i at time n or state j at time n - 1, these are generally dependent on n, correct.

So, the entire quantity, so P ji does not depend on n but the entire this ends up depending on n. So, what are we saying? So, if our Markov chain is running in the forward direction, if I just reverse the tape and run it in the reverse direction I do have a Markov property. The reverse chain is also a Markov chain that is what this equation star says. But if I try to calculate it is transient probability, this transition probability of from i to j ends up being a function of n.

So, what does that mean? So, the reverse Markov chain, it is a Markov chain of course but it is an inhomogeneous Markov chain. So, what are we saying? Even when the forward chain is a homogeneous Markov chain the reverse chain while it is a Markov chain alright it is an inhomogeneous Markov chain in general which is a slightly unfortunate situation. There is a remedy to this, if you assume that the forward Markov chain is started in the stationary distribution.

So, we are now assuming that the forward chain first of all has a stationary distribution pi. And we are starting the forward Markov chain in the stationary distribution at time 0. So, for the forward Markov chain we are starting at the stationary distribution which means the forward Markov chain will always be at pi. Then what happens is that this term out here will not depend on it, because this term will simply become pi j over pi i.

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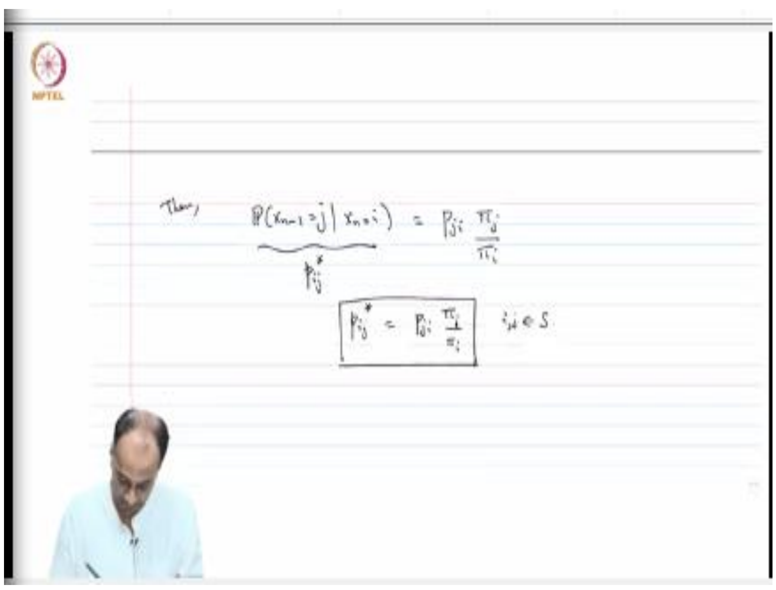

Then, so in this case what happens is that you have P, so let us say then, so all this is assuming that there is a stationary distribution for the formal Markov chain. And I am starting at the stationary distribution then I can write X n = i is simply P ji pi j over pi i. Now this is not a function of n, so this is the transition probability of the reverse Markov chain P ij star. Just to denote that it is not the original chain, it is the reverse chain, I am generating it by a star, P ij star is the probability of transition from i to jin the reverse chain.

That is equal to P ji which is the forward transition probability from j to i times pi j over pi i, where? The stationary distribution is assumed to exist. So, let me just write this down, so this is important, P ij star = P ji times pi j over pi i and this is true for all for any states ij. So, what have we said so far? So, the forward Markov chain is running, if you look at the reverse process also has the Markov property that is what the equation star above says.

But however the reverse Markov chain has a slightly unfortunate property that the reverse Markov chain can be inhomogeneous even when the forward Markov chain is homogeneous. However if you assume that the forward Markov chain has a stationary distribution and that the forward Markov chain is started in the stationary distribution. Then the reverse Markov chain is also homogeneous with the transition probabilities given by this equation.

So, the forward transition probabilities are P ij, reverse transition probabilities are P ij star which are P ji by pi i times pi j. Now that is not what reversibility is, I have not told you what reversibility is. I only told you that a forward Markov chain and reverse Markov chain have 2 different transition probabilities and I have derived the transition probabilities. Now what is reversibility? Reversibility is the property that the reverse Markov chain has the same statistical properties as the forward Markov chain.

That is for any 2 parts of states i and ji P ij star = P ij then such a Markov chain is said to be reversible. So, again reversibility is not that if you run a Markov chain in the, so if you have a Markov chain running like this reversibility is not the property that the reverse Markov chain is also a Markov chain, it always is. And the reverse Markov chain is also homogeneous if the forward Markov chain is in stationary distribution. The reversibility is the property that the reverse Markov chain is statistically identical to the forward Markov chain.