Stochastic Modeling and the Theory of Queues
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Lecture-63
The Birth-Death Markov Chains

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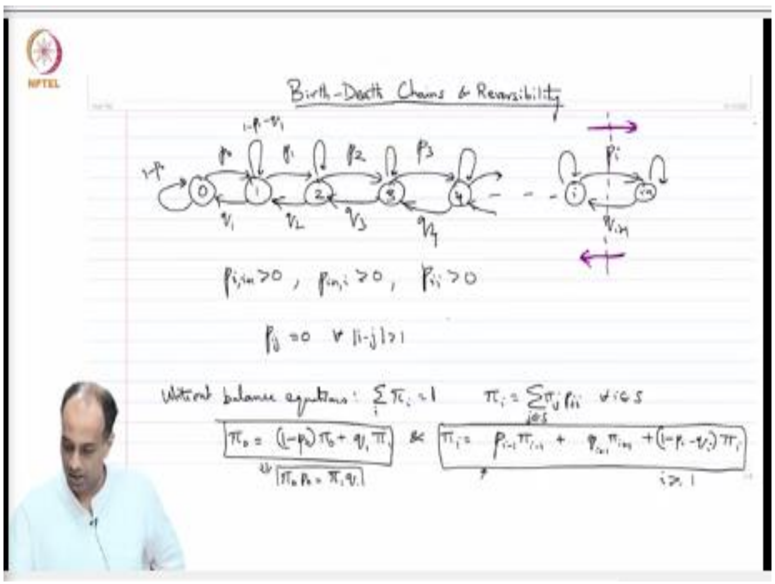

Welcome back, today's lecture we will discuss Birth-Death Markov chains and then our discussion about Birth-Death Markov chains will lead us to reversibility which is a very important concept. We will discuss reversibility and reversible Markov chains. So, Birth-Death chain is a chain that looks like this. Specifically the transitions are allowed only between consecutive states.

So, let us say the state space is 0, 1, 2, 3, dot, dot, dot we are allowed to have transitions only between ii + 1, i + 1, i and self transitions are allowed. For any i P ii + 1 P i + 1 i can be positive, P ii can be positive. But P ij = 0 for all ij that differ by more than 1, so you cannot jump from state 3 to state 1 or state 4 to state 1, not allowed in a Birth-Death chain**.** A Markov chain that has a structure like this is called a Birth-Death Markov chain.

The name is because all these transitions from i to i + 1 can be thought of as a birth in the population and transition from i + 1 to i can be thought of as a death and if the self transition there is no change in the population. So, these Birth-Death chains can be used to analyze populations, they are widely used in analyzing queuing systems, so these Birth-Death processes are very important.

Now this Birth-Death chains if you write out the, now the question is when is this chain positive recurrent? Can we say something about it the steady state behaviour, can we say something about it is stationary distribution and all that? So, all that we have to discuss. So, we are taking the forward jump probabilities as P i's and backwards and probabilities as q i’s. So, if you are in state i the probability of a birth is P i, the probability of a death is q i and 1 - P i - q i is the probability of a self transition into state i.

Because at state 0 the population is 0, so there is no death, birth probability is P 0, that is the model we have. Now for this Markov chain if you write out the, so let us write out balance equations. See what are the balance equations? You should have sum over pi i = 1 and we should have pi i = sum over pi j P ji for all i, these are the balance equations. Now if you write this, so you remember that for this Markov chain P ji is 0 whenever absolute value of i - jis greater than 1.

So, for this system, for this particular Birth-Death chain the balance equation assumes a very simple form. So, if you write out for state 0 you get pi 0 = 1 - P 0 pi 0 + q 1 pi 1. And for all other states i we have pi i = P i – 1, so please note that you can get to state i only from i - 1 or i + 1 or i, because it is a Birth-Death chain. So, we can write this as P i - 1 i pi i - 1 + P i + 1 i pi i + 1 sorry I should write q. So, this P in my notation this is correct but I am notated this as P i - 1 is the probability of both, P i + 1 i is simply q i.

So, there is nothing wrong with what I wrote but I have used the notation q i for the probability of death maybe I should write q i + 1 here actually, q i + 1 pi i + 1 P i - 1 pi i - 1 + 1 - P i - q i, this is the self transition pi i. So, we have these 2 equations, so this is true for pi 0 and this holds this bit holds for i greater than or equal to 1. Now if you simplify this you just get what you get? You get pi 0 P 0 = pi 1 q 1, you can just rearrange this equation to get that.

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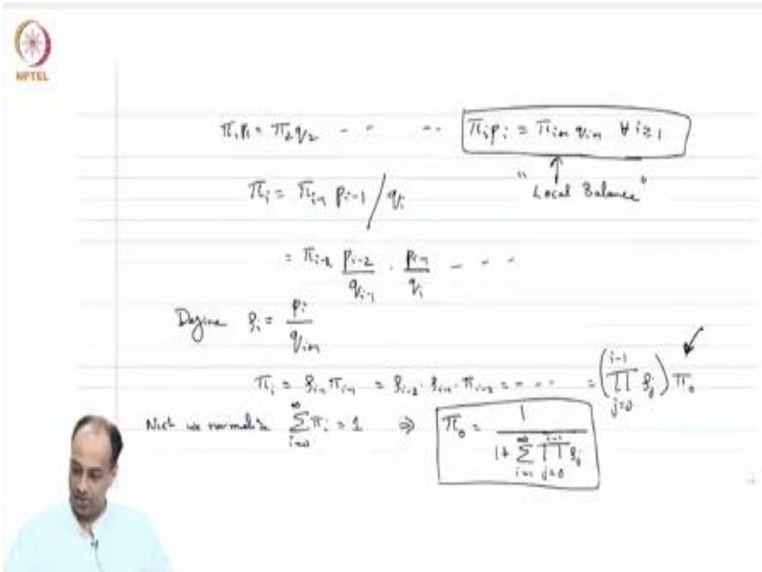

And then if you plug this equation into the equation for pi 1, you will get, so you have an equation for pi 1. So, we can write that equation for pi 1 and you rearrange you get pi 1 P 1 = pi 2 q 2 and so on. So, if you keep doing this you can get pi i q i inductively, you can use induction to show that P i P i = pi i + 1 q i + 1 for all i = 1. So, I am just deriving this, so I wrote out the balance equations and I am simplifying this just using basic algebra.

So, I am getting this form pi i P i = pi i + 1 q i + 1, so that has a nice interpretation. So, if you just look at the sum state i and state i + 1, that is P i and that is q i + 1 because there are self transitions. What we are saying is that, so we are saying pi i P i = pi i + 1 q i + 1. So, if you look at a cut across this graph like this then what we are saying is that pi i P i is the rate of transitions forward.

And pi i + 1 q i + 1 is equal to the rate of transition backwards. So, this pi i P i = pi i + 1 q i + 1 is simply saying that for any 2 pairs of states i and i + 1 the rate of forward transition is equal to the rate of backward transitions. That makes perfect sense; in fact there is an easier way to see this. If you look at any 2 states i and i + 1, the number of transitions and you take any time t. So, for any time the number of transitions that have happened in the forward direction and the number of transitions that have happened is the reverse direction can differ by at most 1, no matter where you start.

That you can never have 2 transitions more in the forward direction than in the reverse direction that is simply not possible in this chain, if you think about it. So, you go forward then you have to come back to go forward again, so that is why the number of forward transitions and the number of reverse transitions can differ by at most 1. Therefore the rate of forward transition should be equal to rate of reverse transitions transition steady state. So, that gives us this equation, now the structure of the chain itself is such that, see this is 1 irreducible chain, so everything communicates.

Also the markup chain is a periodic because there are self transitions. So, the question is when is this positive recurrent and all that? We have to see. Now if you manage to find a pi i, see pi is the steady state the stationary distribution pi i has to satisfy these equations. And if you get a valid probability distributions pi, then we are done, right then we have shown positive recurrence and you have gotten the stationary distribution.

So, in fact you can get all of this from these equations. So, you have if you just proceed with this you get pi i, so if you just look at pi i is simply pi i - 1 P i - 1 divided by q i. So, this is always true, this I can keep on iterating. So, this I can write as pi i - 2 P i - 2 over q i - 1 times P i - 1 over q i dot, dot, dot**.** So, finally this will come out, so if I call let us say let me call rho i as, let us say define rho i = P i over q i + 1.

So, what I am saying is I am taking the probability of the forward transition from i, so the probability of that transition P i over q i + 1 as the ratio to be rho i, this is just some number. So, with that understanding you will simply get pi i = rho i - 1 pi i - 1 which is equal to rho i – 2, rho i - 1 times pi i - 2 and so on and so forth. So, you can finally you will get rho to the i times pi 0, this is true for all i greater than or equal to 1.

These rows are not they are not yet the same, so I should say this is product I beg your pardon. So, this should be rho j = 0 to i – 1, is that correct? So, you just keep on inductively doing this.

So, this you will get rho j times pi 0, I hope this is correct, yeah pi i, this is correct. So, next we normalize, so we want sum over pi i i = 0 to infinity = 1. So, this will give me pi 0 = 1 over 1 + product, this pi is not this pi is PI, this PI is big product, it is not your steady state pi, product j = 0 to i - 1 rho j.

I apologize, I have to write this again 1 + sum over i = 1 to infinity product j = 0 to i - 1 rho j. And so this is your pi 0 and once you know pi 0 you can put this in here to find pi i for any i. Now the question is, is this PI 0 is 1 over something because this big expression? The question is, is the denominator finite? If the denominator is finite then pi 0 will be strictly positive. Otherwise if the denominator is infinite then pi 0 will be 0 then all the pi i's will be 0.

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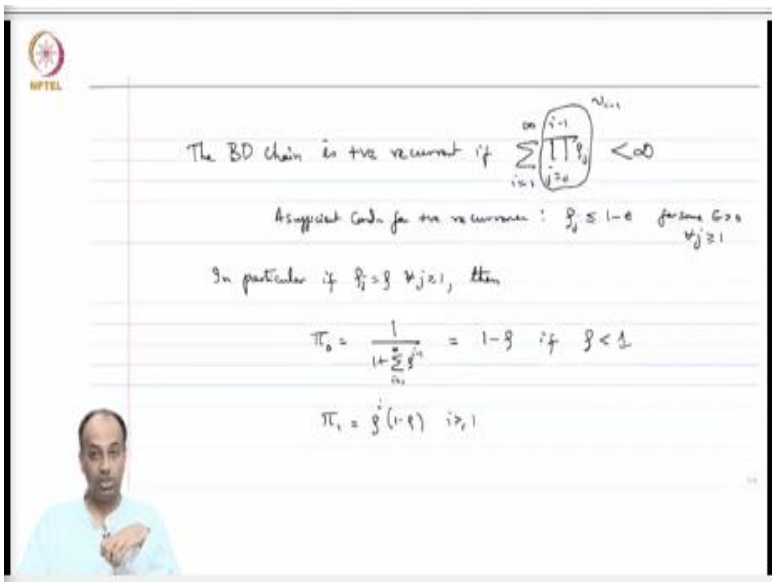

So, the thing is the chain value positive recurrent, the Birth-Death is positive recurrent if denominator is finite. That is if sum i = 1 to infinity, product j = 0 to i - 1 rho j, if this is finite, so this is you can call that guy as sum nu i - 1. So, the sum over nu i - 1 is finite then the Birth-Death chain is positive recurrent. So, sufficient condition for positive recurrence would be for example if rho jis less than or equal to 1 - epsilon for sum epsilon greater than 0 and for all j, then this is a sufficient condition.

Because then this summation would be dominated by a geometric series and you will have a finite sum, there is not necessary condition of course but it is a sufficient condition. In particular if rho j = rho for all j, then phi 0 will simply be 1 over 1 + sum over i = 1 to infinity rho to the i - 1. So, if all the forward transition probabilities and reverse transition probabilities are such that the ratio between them is constant for all i.

For example if forward transition probabilities and reverse transition probabilities are the same in all states then you will have this simple expression. And if this will just turn out to be 1 - rho, it is just a geometric series if rho is strictly less than 1. So, if rho is less than 1 then you have a positive recurrent Birth-Death process. So, pi 0 will simply be 1 - rho and pi i will simply be rho over i times 1 - rho for i greater than or equal to 1. This is because we know that pi i is rho power i times for that rho power i times, yeah this from this equation, rho power i times pi 0, so this is great.

So, usually, so if rho is less than 1 this sort of homogeneous Birth-Death process if the forward probabilities and reverse probabilities are the same, if the rho is less than 1 you have positive difference for sure. Typically when rho = 1 you will have null recurrence and rho greater than 1 you will have transients typically, although if there is self transitions null recurrence is also possible. So, this is something that you can prove, there is an exercise in your book**.** So, rho less than 1 positive recurrence and these are the steady state probabilities.

So, these Birth-Death chains are very nice Markov processes which satisfy this kind of equation, where between any 2 pairs pi i and i + 1 you have the rate of transition from i to i + 1 is equal to the rate of transition from i + 1 to i. And this holds for any 2 pairs of states between which transitions are possible. So, these kinds of balance equations are called local balance equations. This is in contrast to global balance which is usual balance equations which is these, these are called global balance.

And suppose to that this is called local balance where between any 2 pairs of states you have the flow matches in either direction. We will see that whenever this kind of a local balance equation is satisfied between 2 pairs of states, any 2 pairs of states. Then such a process has a certain reversible property, in particular a process like this where the local balance is satisfied will have statistically indistinguishable properties when run forward in time or when run backward in time.

So, these Birth-Death processes are a nice example of reversible Markov chains, Birth-Death Markov chains are reversible Markov chains which we will study next. Of course not all reversible Markov are Birth-Death chains that is not true.