Stochastic Modeling and the Theory of Queues
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Lecture –62
Convergence to Steady State of a Countable-state DTMC (Stochastic Coupling)

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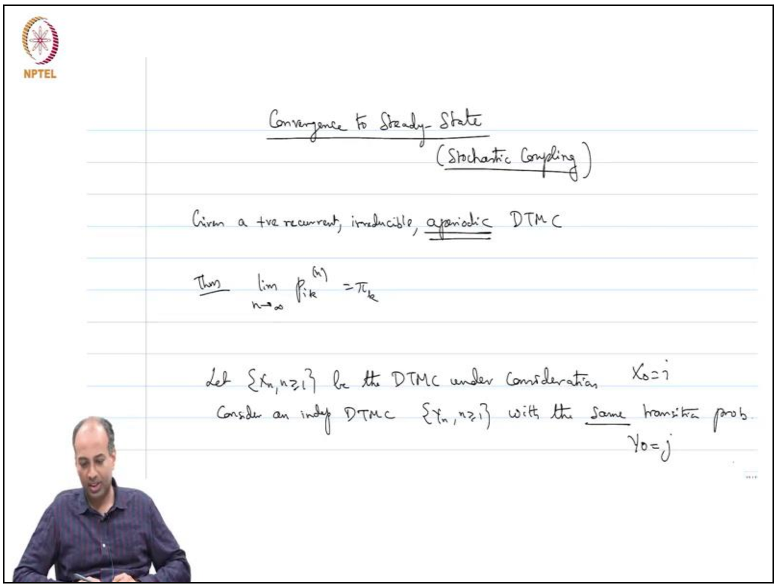

Welcome back in this module we will briefly discuss convergence to steady state. So, I am given a positive recurrent irreducible aperiodic. Aperiodic is now it is necessary, aperiodic DTMC I want to show that limit n tending to infinity P ik n is equal to pi k which is your stationary distribution pi k is uh just as we constructed earlier expected number of visits to k over expected T jj.

Now this we proved for aperiodic irreducible positive recurrent DTMC we proved this convergence using Blackwell's theorem. Now this Blackwell's theorem itself is a really big hammer it is the general proof of Blackwell's theorem is some something that we did not do. So, we have used a theorem a big theorem which we have not proved to prove a relatively easier theorem in about Markov chains.

So, let us remedy that by actually deriving this convergence P ik n is equal to pi k using some elementary arguments. In fact the trick we are going to use is called stochastic coupling this is also from Grimster Zacker section 6.4. So, the basic idea is the following maybe I will not go into all the gory details the idea is this. Let us say be the mark of chain under consideration DTMC under consideration x n is known to be positive recurrent irreducible aperiodic.

And the idea of the stochastic coupling is to consider an independent Markov chain y n with the same transition probabilities. So, in the same sample space; so, i have x n evolving on some on some sample space omega f p on some probability space omega f p. I am starting in state i. So, P ik I want to look at P ik n. So, I have x naught equals i. So, my x Markov chain which is the Markov chain of interest is starting in state i I want to look at the probability of being in state k the x Markov chain being in state k after a very large time n i want to show that P ik n converges to some this pi k .

So, what I am going to do is consider an independent Markov chain on the same probability space with the same transition probability. So, the probability of going from state j to state l is the same for both the x and Markov chain and the y and Markov chain. But I am going to take y naught as some j. So, they are independent Markov chains they have the same exact same transition probability matrix transition probability kernel and I am going to start them off in different states.

So, this stochastic coupling is a standard trick when dealing with Markov chains you basically look for. So, you are starting in two different states and these Markov chains just jump around and evolve and I look at a time the first time when both these Markov chains are in the same state.

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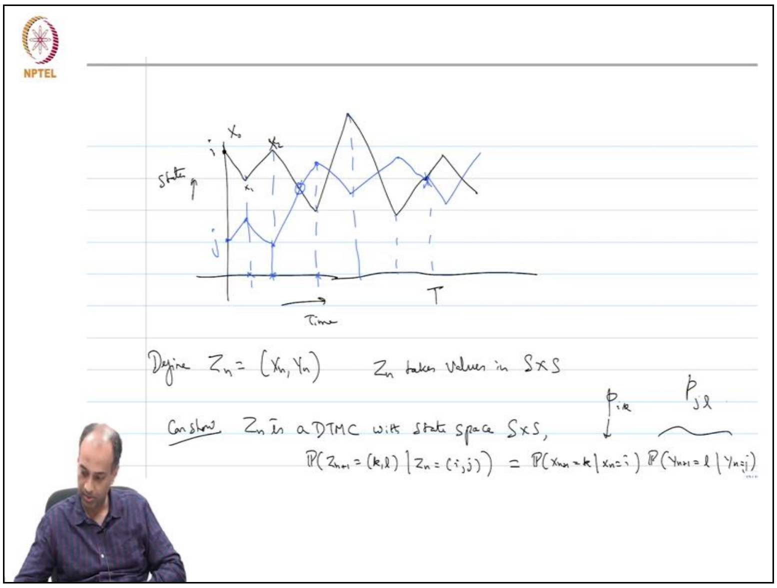

So, to give you a picture say this is all states and this is time my y Markov chain is starting at j it is jumping around and my x Markov chain which I will draw during sorry let us say this is my let us say this is my x Markov chain. So, this is x naught x 1 x 2 and. So, on x naught is j sorry i think i said x naught is i. So, let me say x naught is i and then i say y naught is j and then this guy also jumps around sorry and at this point they are in the same state. So, these are the various so times various integers.

So, at this time they are in the same state the key issue in stochastic coupling is that once these two Markov chain meet they start in different states and they jump around and they meet at some time the first time they meet after that they are stochastically coupled in the sense that the distribution of the further evaluation will be statistically identical of course I am not saying that the blue curve and the black curve will stick together I am not saying that I am saying that the distribution will be the same by the way.

So, the; black curve on the blue curve see that I am looking at only integer time. So, something like that is not a point of I mean it is not you should not look at the chain as having met. So, you should only look at I am just drawing some lines here maybe I should not kind of connect them using these continuous lines but you see you see what I mean this is j this is some other state this is some other state and this is some other state there is actually nothing here. So, this, excuse me.

So, you do not look at these as meeting points this is a real meeting point that is my t the time of first meeting and then of course this black guy goes on and on and this blue guy goes on and on but the point is that I mean they do not have to evolve together but the distribution of the evolution after this time t will be statistically identical that is the whole point of statistics stochastic coupling. So, the argument for this convergence is as follows.

So, x n and y n are these Markov chains. This is so, they have the same state space same transition probability and all that you define z n as the tuple x n y n. So, this takes values in s cross s x n takes values in his y n takes values in s z n takes values in x square s cross s what you can show I will not get into all these proofs is that can show z n is a DTMC with state space s cross s transition probability.

If you look at probability z n + 1 equals k, l given z n is equal to i, j this will simply be probability that x n + 1 equals k given x n equals i and times probability that y n + 1 equals l given y n equals j this is because the Markov chains are independent correct. So, this is nothing but your P ik n that is your P jl.

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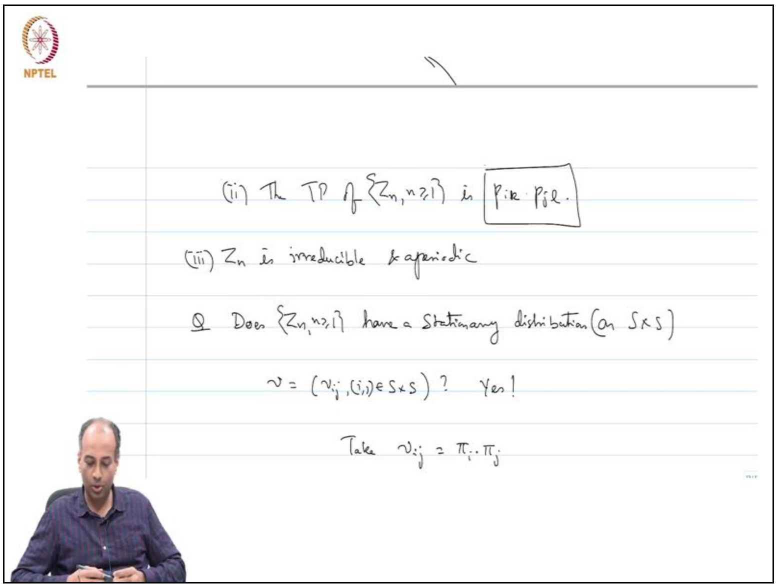

So, the transition probability transition probabilities of z n Markov chain is just P ik times P jl by transition probability I mean this guy we can we can also show that z n is irreducible and a periodic this comes from the irreducible the corresponding properties of x n and y n this requires a proof but it can be shown. So, maybe i should say can be shown state space blah blah blah with that transition probability.

Now question is does z n this Markov chain z n have a stationary distribution on s crosses. So, can I have; so, I want to have nu is equal to nu ij belongs to s i and j belong to s cross s. Another s you take nu ij the steady state probably saw the stationary distribution of the x Markov chain being in i times the y Markov chains being in j you just propose this as a solution. This product of the two stationary distributions you can show that this nu ij is a stationary distribution for the transition probability kernel P ik times p jl.

And for any Markov chain if you do manage to find a stationary distribution we already know that that stationary distribution is unique we also get positive recurrence for free correct. So, this must be unique. So, I am just proposing a solution nu ij stationary distribution for z the z mark of chain as pi a times pi j you can verify that this pi a times phi j satisfies the balance equations for the Markov chain P ik times P jl park of chain z n with transition probabilities P ik P jl. So, this is unique by the theorem that we know and z n is positive recurrent.

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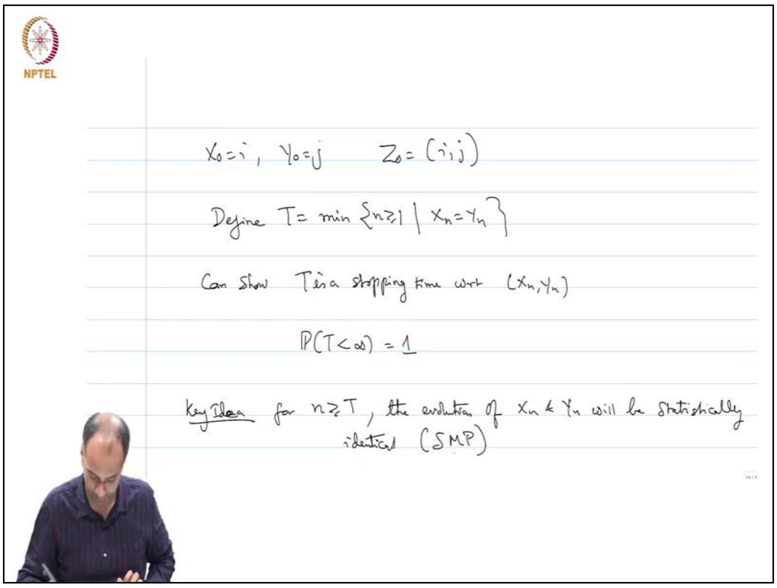

So, we get that for free remember if you the moment you find a stationary distribution that is the only solution to pi is equal to pi P and the chain is positive the current. So, I am proposing u ij equals P ik times pi j as a solution I verify that this satisfies the balance equation for the z Markov chain then the solution must be unique number one number two I get positive recurrence of z for free. Now this is great now the coupling trick works as follows. So, we are taking x naught equal to i y naught equals j.

I am starting in different states. So, z naught is simply i, j. Define T as minimum n such that x n equals y n this is my; so, what am I looking at. So, I let these Markov chains evolve. So, i start from the start the x Markov chain from i, y Markov from z j and i let them evolve and at T they meet T is the first time they meet what you can show is that can show T is a stopping rule stopping time with respect to x n comma y n which is just z n.

And since this you know this is a Markov chain which is positive recurrence that we know to be a positive recurrent we have shown that the z is positive recurrent we can show that T is finite with probability 1. Why is that c i start in state ij z naught is ij I want to get to some state which look like s, s what we can show is that we will get there in finite time because the chain is positive z chain is positively correct.

In fact you can show that expected T is also finite because it is a positive frequency. Now after T the evolution of the Markov chains will be statistically similar. Key idea for n greater than or equal to T the evolution of x n and y n will be statistically identical. So, what I am seeing is that once these guys have met at this point the probability that at any time beyond this T. Let us say at this point the probability that the x Markov chain is in some state l will be the same as the probability that the y Markov chain is in that state.

And this will be true for any time after this stopping time this is because of again strong Markov property you will agree with me that if I start two independent Markov chains at the same state at time zero. Let us say I have two independent Markov chains starting at time zero at the times same state with the identical transition probabilities then you will agree that the distribution of the states at time greater than 0.

The probability of let us say the first Markov chain being in state l will be the same as the probability of being the second Markov chain being the same statement. Let us say at the time T equal to 13 but that is not the scenario here I am not starting at time 0, I am starting at time 0 at different states but at some point they have met and that is a stopping time and because of strong Markov property the further evolution will be statistically identical for these Markov chains this is because of strong Markov property.

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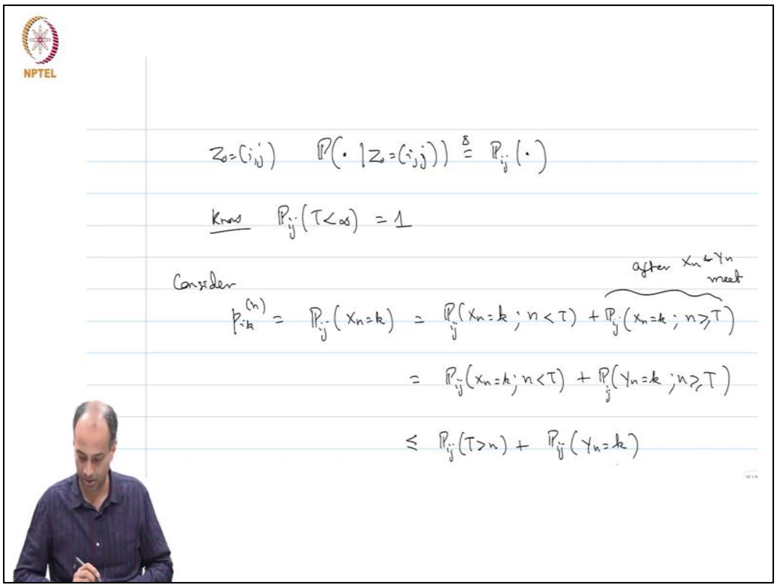

So, with this we can bound we can prove that P ik n goes to pi k the argument is as follows. You look at P ik of n this is the; so, let me look at let me say this is the probability that x n is equal to k given x naught equals i this I can write it as probability that x n equals k t less than or equal to n when x naught equals i plus probability x n equals k greater than n.

So, this is before stopping see by stopping I do not mean that everything stops I just mean before the stopping time and this term is after the stopping where the first term is after stop now the key issue is that this will be same as probability y n equals k T less than or equal to n given. So, this is I can write y naught equals j. So, may be a better way to denote all this. So, maybe I should define you know I will define I will define notation just now let us forget this for a minute.

So, you are starting at z naught equal to ij. So, I am calling I am going to call p dot of given z naught equals i, j as p let me call this P ij of dot. So, this is just convenient notation for me. I know that P ij t less than infinity is 1. Now consider P ik of n which is the which I can write as P ij of x n equals k this is just p ij x n equals k n less than or equal to T + p ij x n equals k semicolon n greater than t n less than t and greater than or equal to t and this bit is after stopping let me say after x n and y n meet.

So, this will just be P ij x n equals k n less than T plus the key idea is this will be P ij y n equals k because after they meet the probability. So, you are looking at n greater than or equal to T after they meet the probability that x n is in state k the same as the probability that y n is in state k because they have the same transition probability and they have already met you are using strong Markov property.

So, this I can say less than or equal to I can just call it P ij T greater than n + P ij I will just write y n equals k.

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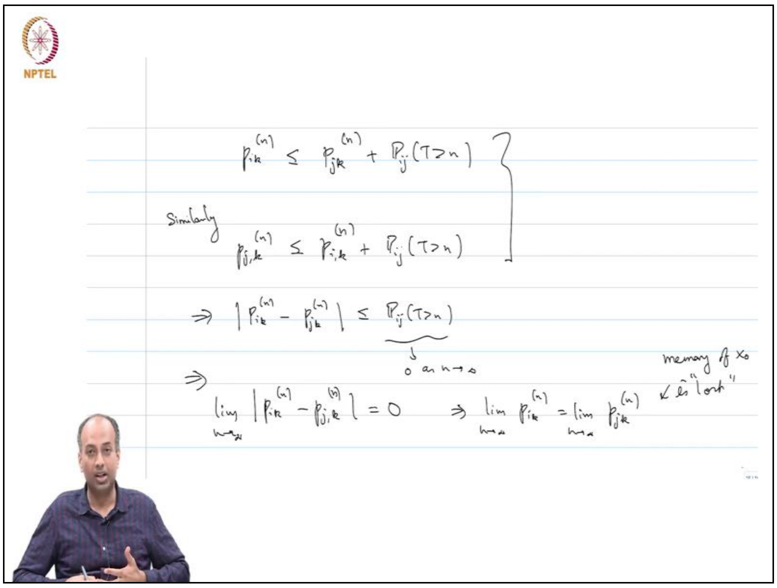

So what have I shown I have shown that P ik n step transition probability is less than or equal to Pp. So, what is P ij y n is equal to k this would be P jk n step transition probability the probability that you are in state case having started at j remember the y Markov chain starts at j P jk n + p ij this probability, probability that T is less than n same. Similarly you can show the exact opposite also you can just interchange the roles of this i and j and simply get P jk n is not equal to P ik n +s P ij T greater than n.

So, this implies the if you look at this difference P ik n - P jk n look at that difference this is less than or equal to P ij T greater than n. This implies now if you send limit n tending to infinity. So, this the hand side will go to 0 as n tends to infinity that is because you are going to this stopping rule it is a legitimate stopping rule you are going to the state the markup chains that is recurrent.

So, you are going to the states are going to meet eventually with probability 1. So, the probability of not stopping until time n the rather the probability that the stopping time is bigger than n goes to 0 as n goes to infinity because T is a legitimate stopping rule this implies limit n tending to infinity P ik n - P jk n is equal to 0 which means that limit n tending to infinity P ik n is equal to limit n tending to infinity P jk n that we could have seen actually even from these two equations you can just you can actually prove that they are equal. So, that is good.

So, what is the show it shows that the probability that is I am in state k after the long time does not depend on where I started the i or j does not matter. So, memory of initial of x naught is lost that is good news but we have to prove that this limit. So, whatever this limit is is has to be so, only a function of k but we have to show that this limit is in fact pi k which is a solution to pi is equal to pi p.

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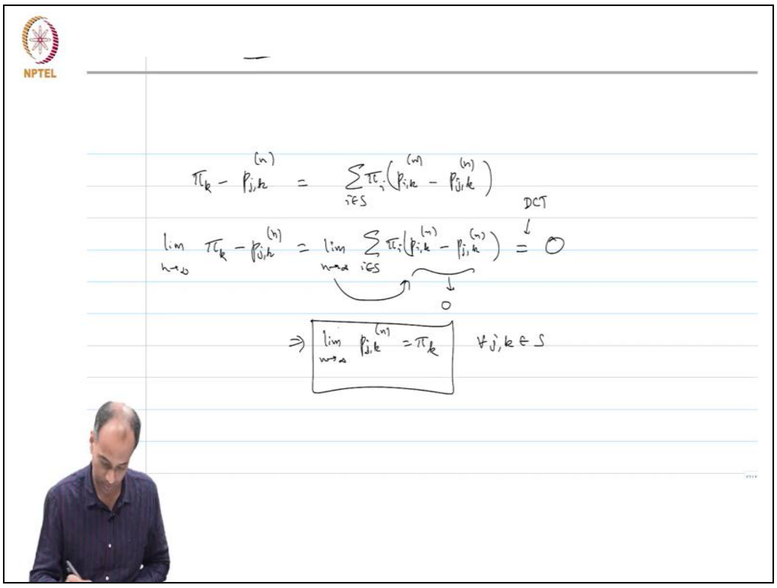

So, that you can argue as follows, so, you just look at pi k - P jk of n. So, if this limit exists. So, sorry so, if this limit exists I know that they have to be equal if they exist. So, I am going to consider this I am going to write this as sum over pi i P ik n – P jk n sum over i. So, this is true because of this if you look at this bits P jk n. So, some over pi i is equal to 1. So, that works out and this is because of the balance equations.

So, this is fairly clear. So, if you take now if you take limit n tending to infinity. So, I have to show that the limit actually exists and it is equal to pi k I have just shown that uh. So, far I have been solid here I have to show that this limit exists if it does exist then the memory of initial states is lost. So, limit anything to into P ik n to infinity pi k - P jk n be equal to limit n tending to infinity sum over i belongs to s pi i P ik of n - P jk of n.

Now each of these terms is going to 0 as n tends to infinity that I have shown because that I have shown here. So, this difference is going to go to 0 but I have a sum which is a potentially infinite sum. So, if i take the limit inside I get 0 but i have an infinite sum outside. So, you need to justify why you can take the limit inside in this case it works out because of dominated convergence you can just say this is 0 this requires some justification.

So, you can write limit n tending to infinity pi k - P kj of n is equal to 0 therefore you have convergence of P jk of n 2 P jk of n equal to pi k this is true for all jk and s. So, it is pretty powerful this technique as you can see I have not used any Blackwell's theorem or any such big hammer like that the main trick is coupling. So, I have considered these two tuples of Markov chains x n y n both independent.

They start in different states I have to prove that the Markov chain z n itself is positive or current by find explicitly finding a stationary distribution. So, I am guaranteed to meet in finite time and after you meet you know these two terms turn out to be equal that is the key step and then it is just real analysis really. So, we have shown that P ij n goes to zero P ij n goes to pi j as n tends to infinity.

So, the periodicity sorry the aperiodicity is used you know you have to use aperiodicity to show that of x n y n to show that z n is irreducible in a periodic and only then without a period is aperiodicity you will not get this result. I will stop here.