Stochastic Modeling and the Theory of Queues
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Lecture –61
Stationary Distribution and the Steady State Behaviour of a Countable-state DTMC - Part 2

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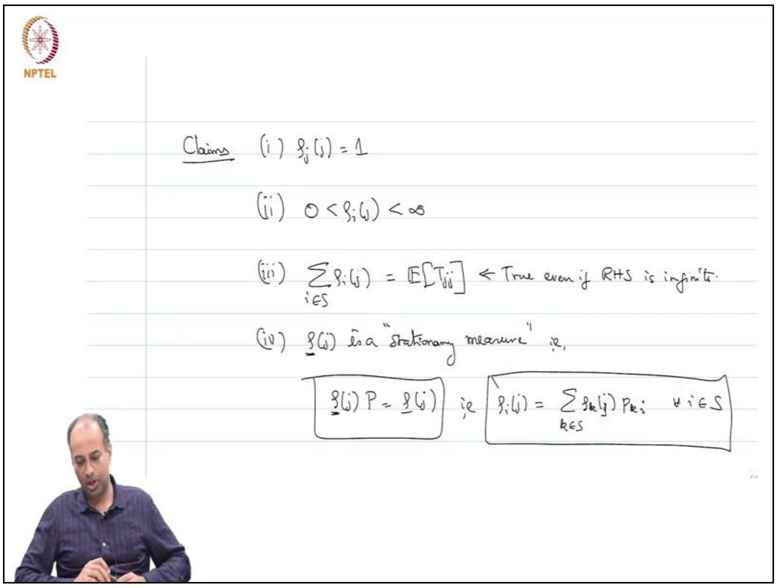

So let me try and indicate you how these results are proved. Proof 3 is easy. So, proof. So, if proof one is very easy. So, let me say 2, 2 is saying that let me prove that rho ij is finite.

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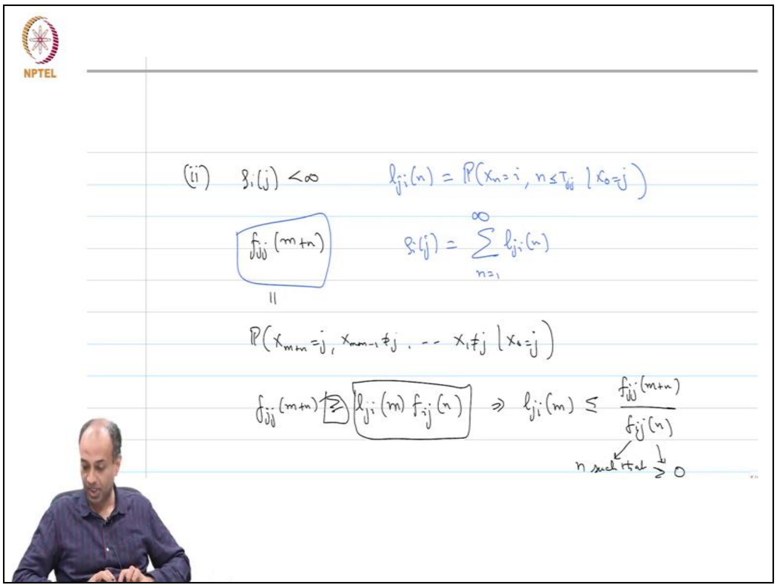

Rho ij is also strictly positive which I will which you can show later rho ij is finite and that can be argued as follows you consider your F jj remember what F jj is? F jj of n is the probability that that you are in state j at time n for the first time after starting in j. So I am going to consider F jj of m + n . So, F jj of m + n is the probability that at time implies n. So, I started at j and a time m + n I came to j for the first time.

This I am going to bound in terms of you know my; these probabilities. So, let me actually I need a one more bit of notation here this probability this guy here I am going to call it l ji of n this whole probability I am calling it as l j of n, l ji of n it is the probability that I am in state i at time n given that there is been no return to j. So, I am going to write. So, let me just rewrite it just l ji of n is equal to probability that x n is equal to i n less than or equal to T jj you have not yet come back to j given x naught equals j. So, just so, that we remember this and we know that rho ij in terms of this l ji is rho i of j is equal to sum over l ji of n from n is equal to one to infinity.

So, now I am going to consider this beast. So, F jj of m + n is the probability if you recall this is the probability that x m + n equals j x m + n minus 1 not equal to j dot dot x 1 not equal to j given x naught equals j. So, this is the probability that I that for the first time I got to state j at time m + n. Now I am going to lower bound this F jj of m + n is greater than or equal to l ji of n l ji of m times F ij of n why is this the case.

So, what is l ji of m. So, I start at m I start at state j and i find myself in state i at time m with before without returning to state j. Again I repeat l ji m is the probability that i find myself in state i at time m but not having return to j then F ij n is the probability that i for the first time return to j after a further time of n. So, the event corresponding to this will certainly lead to the first return to j being a time implement of course there are other ways of returning for the first time from m from j to j.

So, this is a greater than or equal to ok. So, this I think this you will agree this is always true for any m n greater than or equal to one and any state i I can consider. So, this is great. So, I can write F i maybe I should write F. So, l ji of m is. So, l ji I am of m is less than or equal to F jj of m + n over F ij of n and I am dividing because I am taking. So, n is such that this is positive this guy is positive otherwise I cannot be dividing.

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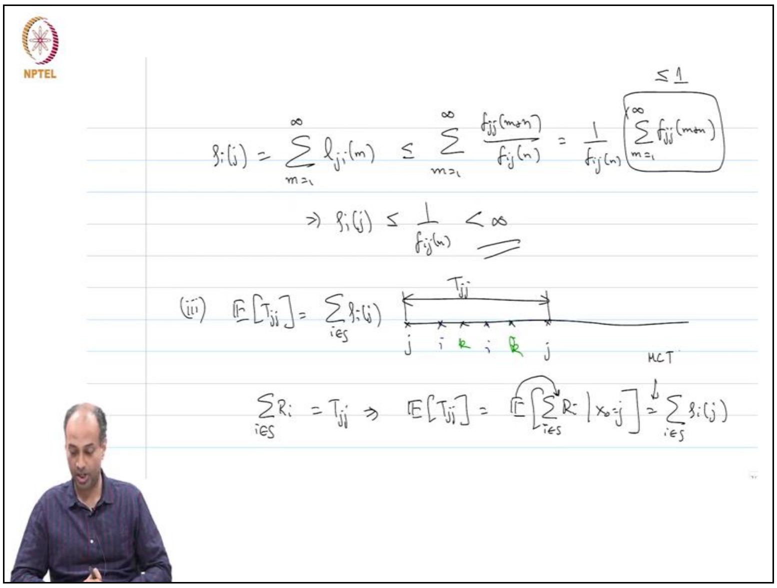

So, if I go back if i just look at rho ij is equal to sum over m equals 1 to 1 to infinity l ji of m which is less than or equal to sum over m equals 1 to infinity F jj of m + n over F ij of n that is one over F ij of n sum over m equals 1 to infinity F jj of m + n. Now this sum is less than or equal to 1 since j is recurrent. So what i have is that rho ij rho if j is less than or equal to 1 over F ij of n and I have taken F ij i have taken n, n such that F ij of n is strictly positive.

And that is always positive that is possible because we are looking at a recurrence state inside one communicating class and this is finite. So, rho ij is finite. I will prove later that rho ij is strictly positive. Next if you want to prove claim 3 which is that expected T jj is equal to sum over i rho i of j sum over i belongs to S. So, it is best done using a picture. So, you are looking at. So, let us say you are at i these times and you're at k during these times k and so on.

In our earlier notation if you look at R i which counts the number of times you are in state i if you look at sum over R i what will that be. So, R i will count that and that k will count that and that. So, if you sum over all the states you are looking at the number of times you are somewhere during this interval which is T jj. So, you have to be somewhere you have to be in one of the states.

So, if you sum you are counting the number of times you are in some state in this interval this has to be equal to T jj correct. So, if you take expectation. So, this implies expected T jj is equal to expected sum over i belongs to s R i all of this is over all of this is given that you start at j but I mean I have I do not have to write this explicitly. So, now if you are allowing that if I take the expectation inside the sum then you will get sum over I belongs to S expected R i given x naught equals j is nothing.

But your row i of j now is this is are you allowed to taking the expectation inside this infinite sum potentially infinite sum it requires a justification in this case all these terms are non-negative and the expected R i we showed is finite out here. So, this is justified by the monotone convergence theorem. So, we are done proving that expected T jj is equal to sum over i belongs to s rho i of j.

So, basically the argument is simple T jj is the sum of all the times sum of the number of times you are in some state. So, T jj is equal to sum over R i and the pushing the expectation inside the sum is justified by monotone convergence theorem that is just a technicality logically it is very easy.

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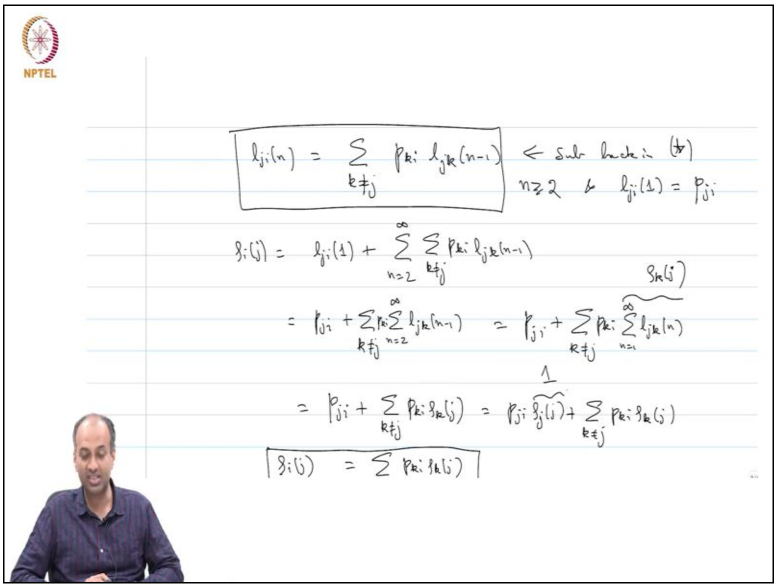

The most important thing I want to show you is for I want to show that rho i of j equals sum over k belongs to S rho k of j P ki this is true for all I how do I prove this. So, remember that rho i of j recall is nothing but sum over n equals 1 to infinity l ji of n where you know I am just writing that again l ji of n is defined like. So, now I am going to condition on so, this this is nothing but sum over n equals 1 to infinity let me write this a probability that x n equals i n less than or equal to T jj given x naught equals j.

Now I am going to write this out with one more level of conditioning this is your del j of n let me call this star now l ji of n is equal to probability that x n equals i. So, I am going to say x n minus 1 equals k ok n less than or equal to T jj given x naught equals j and I am going to condition I am going to basically sum over all the k where k is not equal to j. So what am I doing I am just using theorem of total probability.

I am just looking at all the possible values that x n - 1 can take x n - 1 cannot take the value j because I am insisting that t less than or equal to T ji have not yet returned to j so I am just going summing over all the other states not including j this just turns out to be sum over k naught equals j probability x n is equal to i given x n minus 1 equals k n less than or equal to T jj x naught equals j times probability that x n minus 1 equals k n less than or equal to T jj given x naught equals j.

I have just written out this in terms of theorem of theorem of total probability and used conditional conditioning. Now if you look at this, this is the probability that x n equals i given x n - 1 equals k x naught equals j and something to do with this stopping rule T jj. Something to do with this T jj being bigger than or equal to n. So, we can invoke strong Markov property because of strong Markov property plus this term will be equal to P ki.

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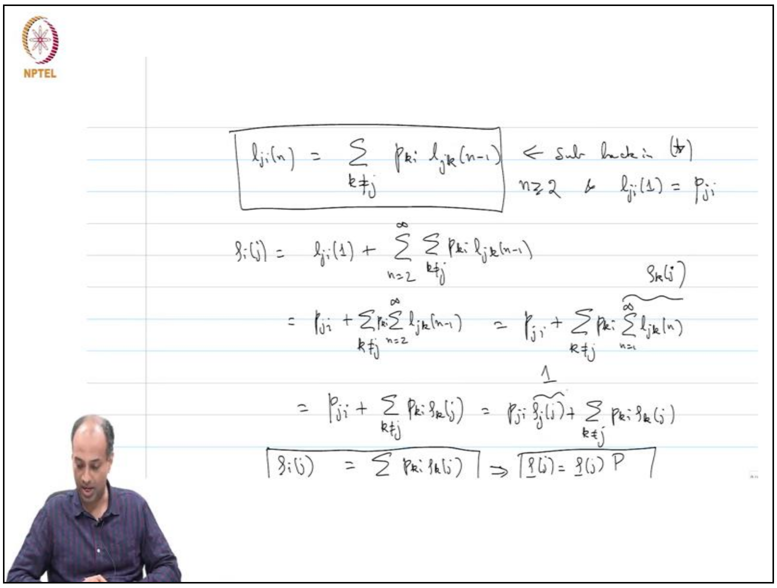

So, this will be equal to sum over k naught equals j P ki and now what is that that will just turn out to be l j k of n - 1 . So, the probability that I find myself in k at time n - 1 this you can this involves strong Markov property and then arguing that you have not returned to j. So, this will just turn out to be equal to l j k of n - 1. So, that is good. So, I am saying that. So, what I have shown is that l ji of n is equal to that.

So, this I substitute back in star substitute sub back in star what do I get star is that guy rho ij equal to sum over n equals 1 to infinity. So, I want to take l ji of 1 outside plus I want to sum from n equals 2 to infinity. So, this by the way if you look at this if you look at this equation which is very useful this is valid for n greater than or equal to 2. And what is l ji of 1 lj y of 1 is equal to simply P ji.

It is the probability that i find myself in state i given that i started in state j l ji I have to go to state i at time 1 given that i started at state j which is simply p ji . So, I am taking the term out lji of 1 + sum n is equal to 2 to infinity

I am going to sub this back l ji of n. So, I am going to write sum over k naught equals j P ki l jk of n -1 what is what is l j of 1? I know this is equal to p ji plus; if you will if you will allow me to interchange the orders of summation because everything is non-negative. So, I will write sum over k naught equals j sum over n equals 2 to infinity. So, I will just write P ki P ki l jk of n – 1. What is this sum this is nothing but, so, I can just maybe I should just write one more step I should probably just write P never mind this P ji plus sum over k naught equals j P ki sum over n equals 1 to infinity l jk of n.

And this we know to be equal to rho k of j we know that term to be equal to rho k of j so that so, that is very happy for us. So, this whole thing becomes P ji + sum over k naught equals j P ki rho k of j but I have one i know one thing that i know rho jj is equal to one this is claim one. So, that I can write this as P ji rho j of j + sum over k naught equals j P ki rho k of j that is because that term is equal to 1 I already know that.

So, this then you get what you want this just turns out to be sum over all k some over all states. So, I have rho ij equals some over all these states P ki rho k of j and that is really that's just saying that. So, this is basically saying that rho of j is equal to rho of j P. This is true for all i. This is this equation is true for all it belongs to S. So, I get this balance equation for this rho's.

So, what have we shown have shown all four properties except I think I have to show that remember there is rho i of j I have to show to be strictly positive.

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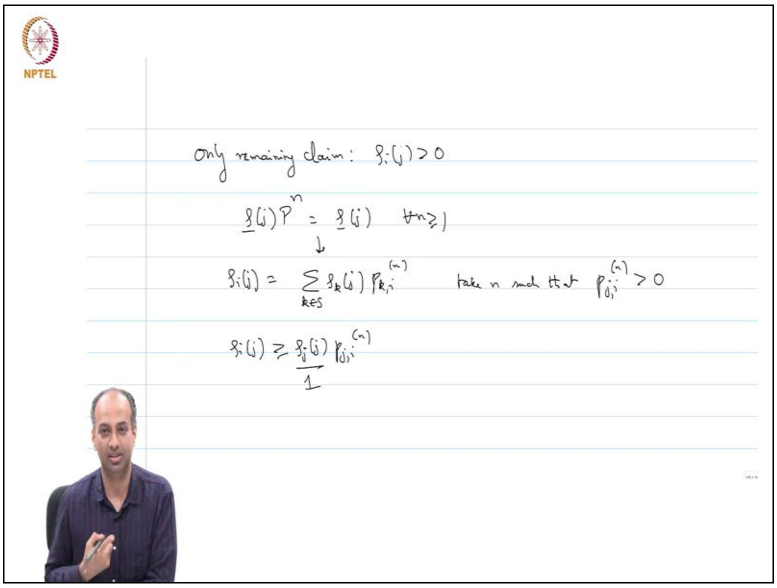

So the only remaining claim is rho i of j is strictly positive ok for that we argue as follows you have rho of j P nth power is equal to rho of j for all n greater than or equal to 1 correct that is because rho j we showed it to be a stationary. We know we showed that rho j p is equal to rho. So, rho j p square is equal to rho and so on. So, we can use this cleverly. So, what I will do is I will write rho i of j is equal to sum over rho k of j P ki n, n step transition probability sum over all k I am just writing this in explicit form.

So, this P, so, there exists n take n such that P ji n is strictly positive then I will get rho i of j is greater than or equal to rho j of j P ji n. Now rho jj is what rho jj is simply equal to 1 and P ji of n is strictly positive for some n you take such an n this is since this is true for all n you can take such an n for which this is strictly positive. So, you can just strictly positive. So I have proved everything I wanted to prove you go back to all these claims uh.

So, this was easy this i proved in two parts and this was also easy basically just monotone convergence theorem and this was my main result. So what is the summary of what I have said. So, far I have basically constructed this rho ij as the expected number of which is to i between successive basis to j. This has some nice properties it is a stationary measure. So, it is strictly between 0 and infinity and it solves the equation x p is equal to x.

So what we have shown by explicit construction we have explicitly constructed a solution to x p is equal to x and we have interpreted this solution. And furthermore if the Markov chain is positive recurrent you can define divide this rho i of j divided by expected of T jj which is finite for a positive equal chain you can get a stationary distribution. So, for a positive recurrent DTMC we have shown that we have shown an explicit construction of pi.

So, pi i is simply rho ij divided by expected T jj where the expected number of visits to state i between visits to j divided by expected recurrence time of state j. So, this is an explicit construction I invite you to go over Grimster Zaker statement of this, this is essentially it and then they go on to prove uniqueness up to a multiplicative constant. And show that if so, xp essentially Grimster shows that xp is equal to x has a unique positive solution.

Unique being unique up to a multiplicative constant and if sum over all these x size is infinite the chain is null recurrent and the sum over x i is finite the chain is positive recurrent and you can construct a stationary distribution among states. So, this I invite you to go through in section 6.4 of Grimster Zaker. I will stop this module here.