Stochastic Modeling and the Theory of Queues
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Lecture –59
Stationary Distribution of a Countable State Space DTMC and Renewal Theory (Contd.,)

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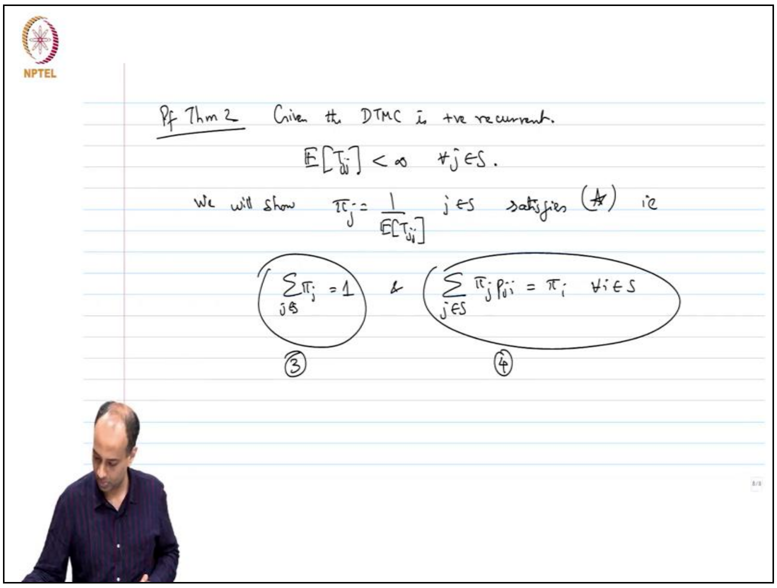

Now I have to prove theorem two. What does theorem two say? Theorem two says that if the chain is positive or current then this then there exists a solution to pi is equal to pi p. So, given the irreducible DTMC is positive recurrent. So, therefore I can take expected T jj is finite for all j because I am given there is positive recurrent. Now I have to show that pi is equal to pi P has a solution what we will show pi j is equal to one over expected T jj for j belongs to S satisfies star ie we should have sum over pi j equal to 1 and sum over j pi j P j i is equal to pi i this is what star is.

So, what am I doing I am proposing a solution one over expected T jj i mean this is this I know from the this is what I know to expect I am just simply going to show that this satisfies the balance equation. Now I just have to show that I propose the solution I just have to show that these two solutions these two equations are valid. Now let me first show that let me show this is 3 let me just show that is 4.

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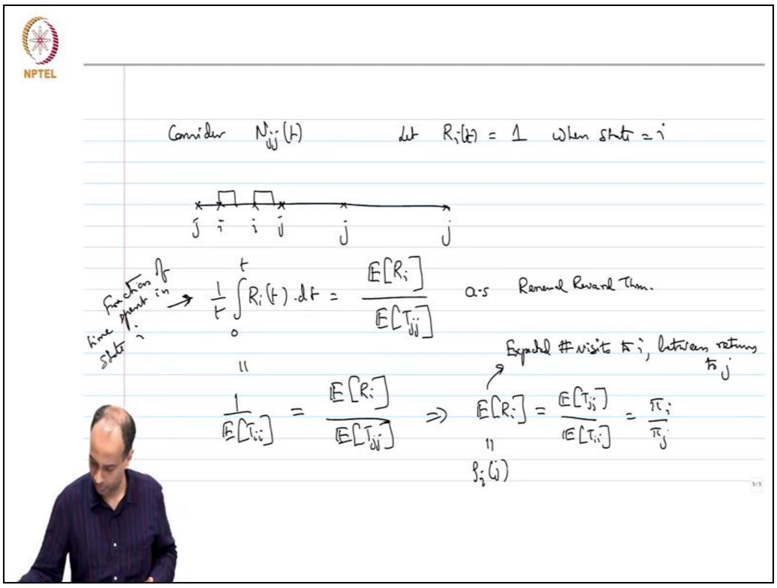

So, we will use renewal reward theory consider the process N jj of t starting at state j and i am going to count the number of times i hit state i in each renewal interval. See I know now that see I am given in theorem 2 I am given that the chain is positive recurrent. So, I know that N jj t is a renewal process. So, let Ri of t. So, is I am going to you know define a reward process which is one when state is equal to i then I know that one over t integral 0 to t Ri of t dt is equal to expected Ri which is the expected number of times i will be in state i between successive which is to state j over expected T jj.

So, this is almost surely this is by renewal reward we already saw this equation. So, if you look at this is the fraction of time spent in state i and this we already know to be equal to what. So, if you look at the process N ji of t it will be the by strong law for the delay annual processes we can say that this is equal to one over expected T ii. So, I have one over expected T ii is equal to expected Ri over expected T jj correct. So, this expected Ri is what.

So, this implies let me just write down expected Ri is therefore equal to expected jj over expected T ii which is simply pi over pi j. So, expected Ri is the expected number of visits to i between successive returns to j. So, expected Ri I am going to call it I am going to denote it by rho i of j this is going to be useful later.

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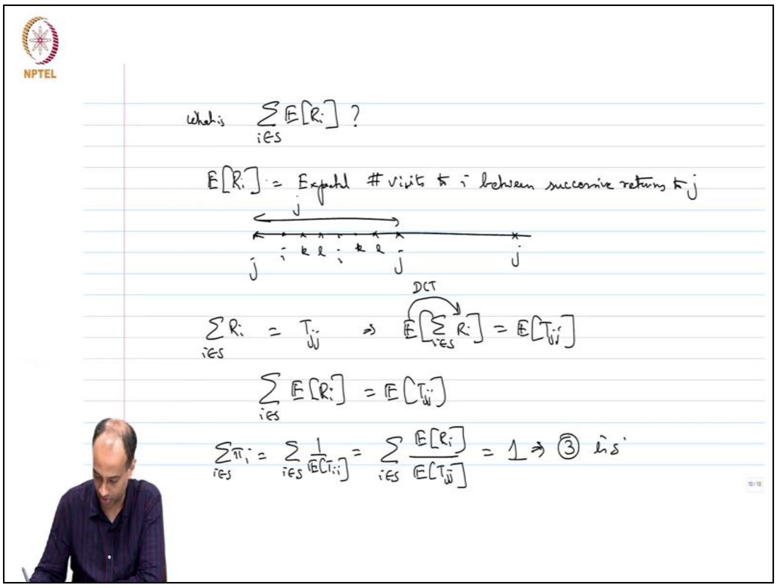

So, this is good expected number of visits to i between results returns to j satisfies pi i over pi j. Now what is sum over i belongs to S expected Ri. So, I let me write. So, this is I am essentially trying to sum I am trying to show that some over pi i is equal to 1. So, I am just going to go back to this equation this equation. So, expected Ri this expected number of visits to i between successive returns to j.

So, if you look at this process j, j, j how many times do I hit i. Now if I take sum over i belongs to S Ri that is that will be equal to the number of returns to state i plus the number of returns to state k. So, number of visits to state k plus number of returns to state L and so on. So, I am summing over all the states I am going to number of visits to all the states between this is to ji I am going to sum this over all j since the so, all i sorry.

So, this has to be equal to T jj because the process has to visit some state or the other. So, this implies expected sum over i belongs to S Ri which is equal to expected T jj. So, if you are willing to move this expectation inside this infinite summation which is justified by Dominated Convergence theorem then you can argue that sum over i expected Ri. Expected Ri is the expected number of which is to i between successive which is to j if you sum it over all states this has to be equal to expected T jj which means that if i go back.

So, I know this. So, I know sum over pi i which is just sum over i belongs to S one over expected T ii is equal to sum over i belongs to S expected Ri over expected T jj but some over expected Ri is simply expected T jj. So, that is equal to one. So, we have shown that sum over pi i is equal to 1. So, this is 3 is shown next i want to show 4.

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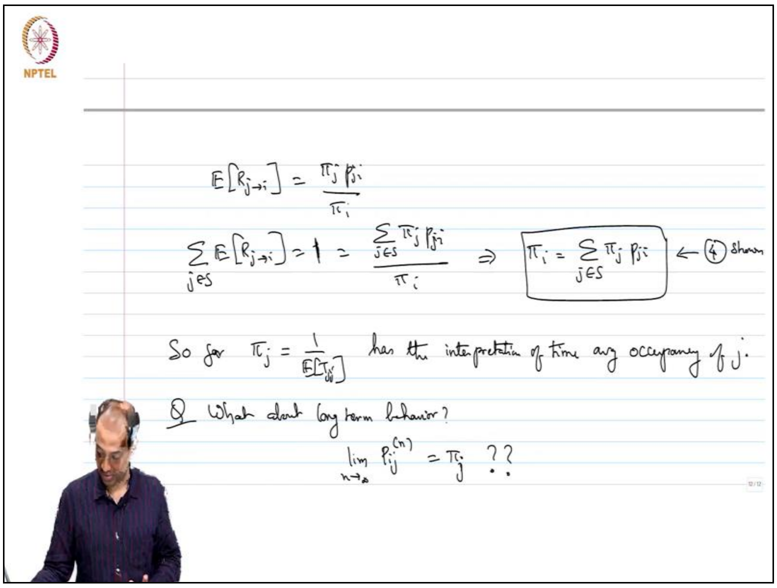

Next I want to show 4. This 4th is saying that sum over pi j P ji is equal to pi. The proof is somewhat similar I will just indicate the word process that we use so let me call this R j arrow i. So, reward process whenever the chain transitions from j to i. So, now I am going to let us say look at state i I am going to look at i and I am going to look at a reward which is one whenever I have a transition from j to i.

So, if I look at this process N ii of t. So, this transition may go from k to i this transition may go from j to i. So, in this case I will accumulate one reward here I will not accumulate any reward only when I go from j to i do I accumulate a what. Now if you look at this again if you look at sum over R j arrow i sum over j belongs to S this will be equal to 1 in each renewal interval that why because the before you hit i in each annual interval you have to transition from some j or the other.

So, again I can argue that sum over j belongs to S expected R j arrow i is equal to 1 this is again I am pushing the expectation inside the sum and all that which is justified. Now but what is the average rate at which this reward occurs this rate of this reward time average reward rate equals. So, I have to be in state j and then make a transition from j to i. Now what is the time average reward rate it is the time average rate of being in state j and then making a transition from j to i.

So, what is the time average fraction state j 1 over expected T jj then I have to make a transition from j to i the probability of which is P ji but since I have taken one over expected

T jj as pi j this will simply g simply be pi j P ji this is the time average bar rate this is nothing but for the renewal process in a N ii is expected R j to i transition over expected T ii this is just by renewal robot theory this is nothing but if you take this guy here let me I should write this as pi i expected R j to i.

So, the expected reward for a transition to write over a renewal interval for a transition from j to i is simply pi j P ji over pi i but I know that if I sum this over j belongs to S I have expected R j to i this must be equal to one. It is because of that that must be equal to sum over j belongs to S pi j P ji over pi i implies pi i is equal to sum over j belongs to S pi j P ji. So, we have shown 4 also. So, we have proved theorem two what have we proved.

So, given a positive recurrent irreducible DTMC I know that expected T jj is finite I have shown that 1 over expected T jj you call that pi j satisfies the balance equations. So, whenever you have a positive recurrent Markov chain 1 over expected T jj has the interpretation of this stationary probability in terms of time average occupancy of state j. So, to summarize, so, far we have shown that given first theorem one showed that if pi is equal to pi P has a probability distribution solution then that chain is automatically positive recurrent.

So, if sum by some means you manage to find the solution to pi equal to pi P then that chain is positive recurrent the solution you found this unique in the sense that the pi j will be equal to one over expected T jj. And conversely if the chain is the current positive recurrent then one over expected T jj has the interpretation of the solution to pi is equal to pi P throughout we have interpreted pi j as time average occupancy of state j. So, far. So, that is true regardless of any considerations of periodicity.

Now I want to look at i want to briefly look at similarly i say this. So, far pi j which is just one over expected T jj has the interpretation of time average occupancy of state j. Now I am not just satisfied with the fraction of time spent in state j. I want to know if I run the Markov chain for a very long time do I have convergence of the probability of being in state j to be equal to pi j that I have not shown.

So, far I mean that I have not about not even talked about everything I have done. So, far is time averages I have only used time average renewal reward process. I have just said if the state is state j is positive the current is equivalent to saying that 1 over expected T jj which is pi j has is a solution to pi is equal to pi P and therefore pi j has the interpretation of the fraction of time spent in state j.

What about question what about long term behaviour. So, can I talk about things like if I start in any state i can I say that limit n tending to infinity P ij n is equal to pi j big question mark it is not at all clear that this can be done. So, that is what we will address next steady state interpretation.

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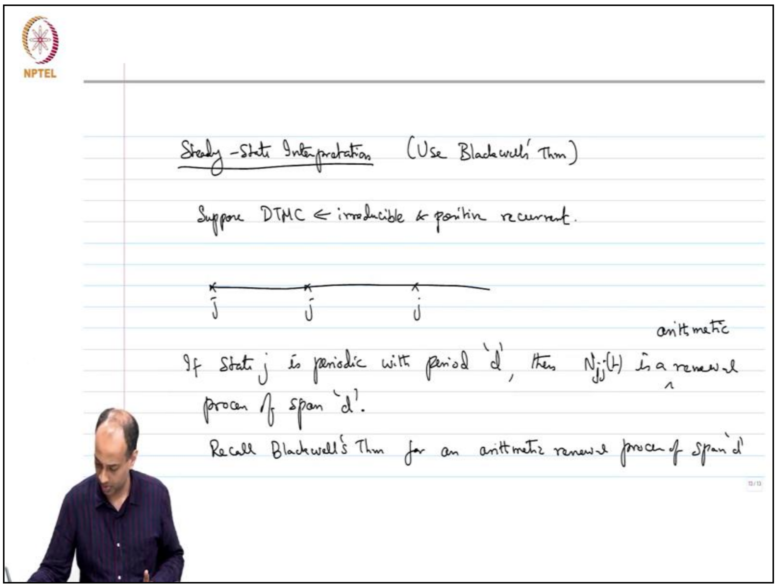

So, we are going to now interpret pi j as a steady state probability not just a fraction of time spent in state j. So, this pi j you know it is on the one hand it is a stationary distribution. So, you start in pi j you remain in pi j all also because of Renewal Reward theory we have shown that pi j has the interpretation of the average fraction of time spent in state j. Now I am going to go into a different interpretation which is a steady state interpretation.

Can I say that no matter where I start the probability that I will be in state j after a very long time n approaches pi j can I say that answer is not always but sometimes in some nice cases you can say this and we are going to use Blackwell's theorem again from Renewal theory. Now we are going we have to have periodicity considerations. So, because we have a periodic chain then you may not have convergence of P ij.

Suppose we have a DTMC which is irreducible and positive recurrent. Let me look at this renewal process of returning to j. If state j is periodic with period d then N jj t see this if you are saying that returns to j can only occur at multiples of j I am saying that if j is periodic with period d it we are saying that returns to j can occur only at multiples of d. Then N jj t is a renewal process of course.

Renewal process of span d is an arithmetic renewal process all the arithmetic renewal process. It is an arithmetic renewal process of span d because all this returns to state j only occur in multiples of times d. Now recall Blackwell theorem for arithmetic renewal process of span d.

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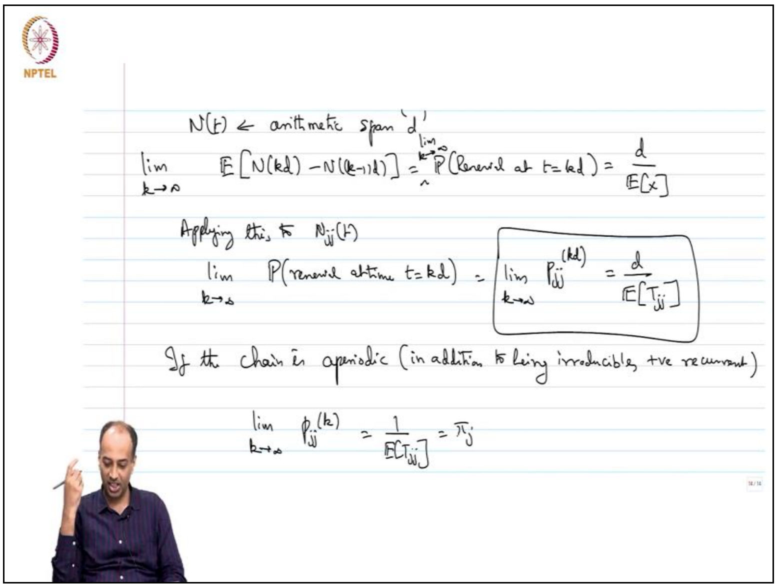

It just says that let us say n of t is it is an arithmetic annual process of span d then we have expected number of renewals so, N of t + kd - N of t plus also maybe I should write it as n of kd - n of k - 1 d all is the problem which is the probability of renewal at t is equal to kd is nothing but d over expectation of x where x is the you know the renewable distribution this we know from Blackwell.

So, for this Markov chain, so, apply this to a to N jj t which is an arithmetic renewal process of span d we have probability of renewal it at time t is equal to kd. So, this is of course this says limit. So, this is not for all k this is S limit k tending to infinity. So, limit, so, everywhere I should have limit k training to infinity here also I should have limit k tending to infinity this is equal to limit k tending to infinity P jj at time kd.

This I can say is d over expectation of T jj all this is by black with now you say if the chain is a periodic of course in addition to being irreducible and positive recurrent. So, if you have a countable state DTMC which is irreducible positive recurrent and a periodic then that is just d equal to 1 isn't it. So, i can write limit k tending to infinity now you have a arithmetic renewal process of span one the span d is simply one now.

So, this is k negative infinity P jj of k. So, this is the k step transition probability of. So, starting at j what is the probability that I have been j after time k and I am looking at this after after a very long time this is equal to 1 over expected T jj which is simply pi j. So, given that I started j the probability that I find myself at j after a very long time is by Blackwell pi j which means that I have this interpretation of finding myself this pi j is steady state interpretation is valid.

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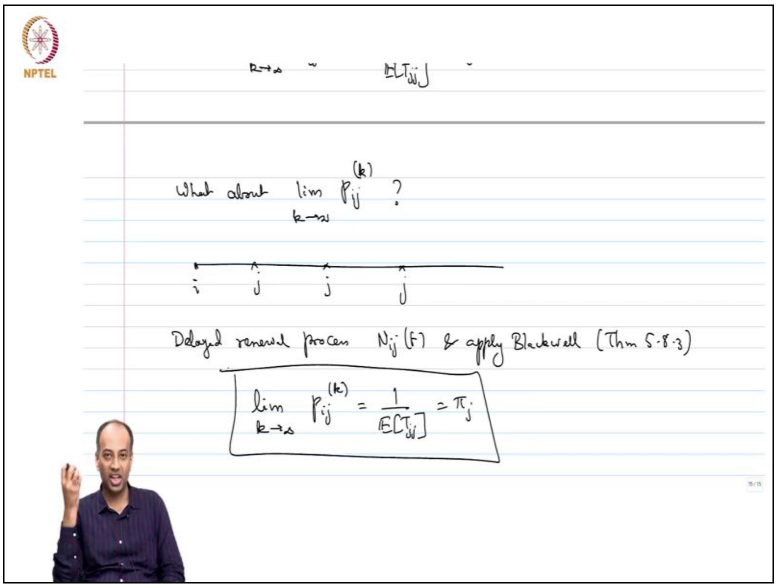

But this is only for starting state j what about limit k tending to infinity P ij k. We can consider we are starting at i then I am looking at returns to j. So, you can look at the delayed renewal process look at the delayed annual process N ij of t and apply Blackwell. So, remember that there is a corresponding Blackwell theorem even for delayed annual processes, let me just find out for you this is theorem 583 in Gallagher.

You can do that and say that limit n tending to infinity and tending to infinity sorry k tending to infinity P ij k is equal to 1 over expected T jj which is equal to pi j for any state i. So, now because of Blackwell thanks to Blackwell I have an interpretation of P ij k after very long time k no matter where I start the probability I find myself in state j after a very long time is equal to pi j limit k tending to infinity P ij k step transition probability is equal to pi j this is from Blackwell.

So, you can see how powerful some of these renewal results are renewable reward theory was heavily used. Now we are using the powerful Blackwell's theorem to give P ij k to show that P ij k has a limit as k tends to infinity for a periodic Markov chain and pi j has the interpretation as the limit of P ij k as k tends to infinity. So, this is quite powerful as you can see the new renewal theory, renewal reward theory and Blackwell's theorem.

And it helps us derive these fundamental results about countable state DTMCs all of course this is not the only way to show these long term properties we have taken the approach of renewal processes there are other approaches as well. For example there is an approach of constructing the steady state probabilities pi j directly by looking at the expected number of visits to a state i between successive which is to state j.

Maybe I will briefly mention that a little bit later but this is one valid way to approach it there are other ways to approach these Markov chains as well. I will stop here.