Stochastic Modeling and the Theory of Queues
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Lecture –57
Renewal Theory applied to DTMC's

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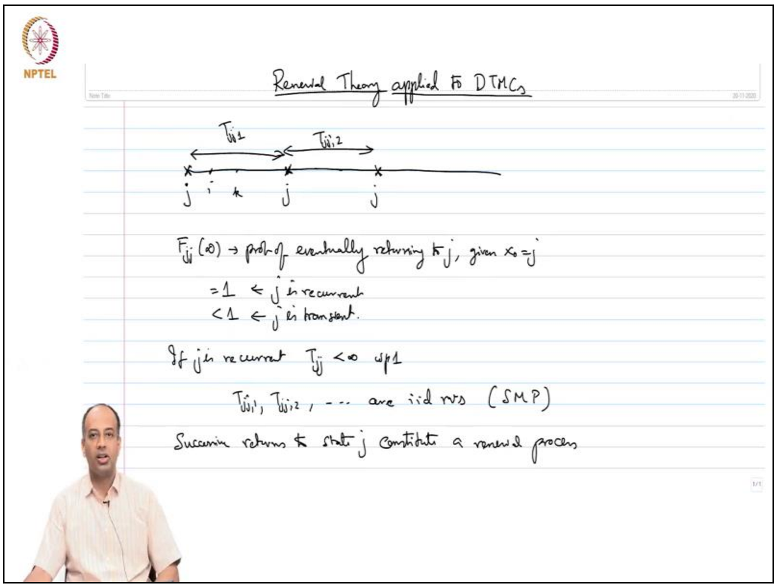

Welcome back we were looking at the process of returns to state j say let us say the Markov chain starts at state j at time 0 and then it visits a bunch of states and then returns to state j at some point and so, forth. We looked at F jj infinity which is the probability of eventually returning to j given that we started j. So, if this F jj is equal to 1 then we said j is recurrent and it is less than 1. So, j is transient.

Then so, in the case when if j is recurrent then this T jj is less than infinity with probability one and then we said that if you look at these times of subsequent returns to j T jj 1 T jj 2 and. So, on T jj1 T jj2 are iid this is from the strong Markov property. We said that if you start at j and first time you return to j the subsequent evaluation after T jj1 also satisfies the same transition probabilities and this is the strong Markov property that we studied in the last module. Using this we can straight away get some results that we know from renewal theory.

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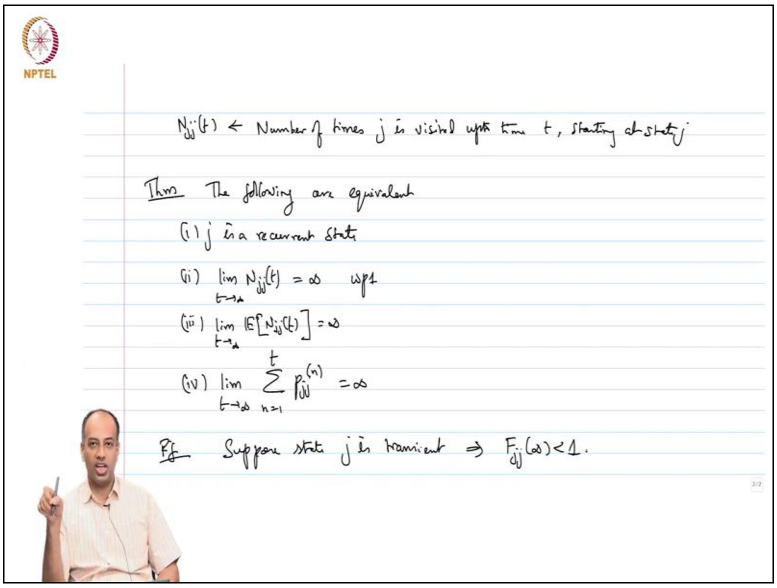

So, let us do that lets let me say N jj t is the number of times j is visited in; so, up to time t, time t included. And we can already state this theorem starting at j. So, let me say starting at state j the following are equivalent j is a recurrent state number 2 limit t tending to infinity N jj of t is infinity with probability one limit t tending to infinity expected N jj of t is equal to infinity.

Number 4 the emit t tending to infinity sum over N is equal to 1 to t P jj N is equal to infinity. So, this theorem says that the following statements are equivalent that is j is a recurrent state if and only if the number of returns to state j until time t goes to infinity as t goes to infinity with probability one and the expected number of this is to state j at until time t also goes to infinity and finally the sum of these probabilities goes to infinity.

We are saying that all 3 all these 4 statements are equivalent all e any one of this implies the other vice versa all proof. So, this is fairly easy suppose at least the first 3 are fairly easy. Suppose stay j is transient ok this implies that this by definition is saying that F jj infinity is less than one all. So, the probability of starting at j and eventually returning to j is strictly less than one.

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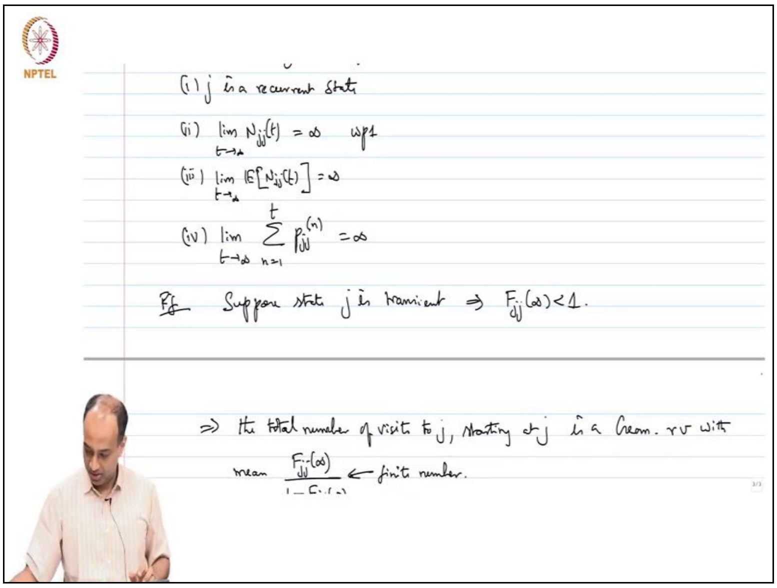

This implies that the total number of visits to j starting at j is a geometric random variable with mean F j over 1 - F jj this is because you have you have a positive probability namely of 1 - F jj of never coming back to j all. So, if you start at j and come back to j the probability of doing that is F jj infinity and that is strictly less than 1 and you can the number of times this will happen is simply the number of times this if you know this it's as though you are tossing this coin with probability F jj infinity.

So, if it fails with which happens with 1 - F jj infinity probably 1 - F jj infinity then you will never return to j. So, the number of times you will return to j can be shown to be a geometric random variable all with finite mean. So, which means that the number of returns to j starting at j and the expected number of returns today starting at j are both finite this guy is a finite number all. So, if j is transient the number of eventual number of returns to j as well as the; expected number of eventual returns to j are finite.

So, this implies that for a recurrent state. So, this establishes that if the number of returns to stage j starting at j and that expected number of returns is infinite then the state j must be recurrent because if its transient we know that the expected number of returns to state j is finite and the number of returns is also finite with probability one therefore if this number is infinite then the number of returns to j must be then j then state j must be recurrent all.

So, that establishes the implications of 3 and 4 sorry the 2 and 3 implying that state j is recurrent is clear if state j is recurrent of course we know that N jj t is a renewal process and therefore we know that limit t tending to infinity N jj t is infinite with probability one and expected N jj t also goes to infinity as t tends to infinity. So, we can easily argue that the first 3 statements are equivalent.

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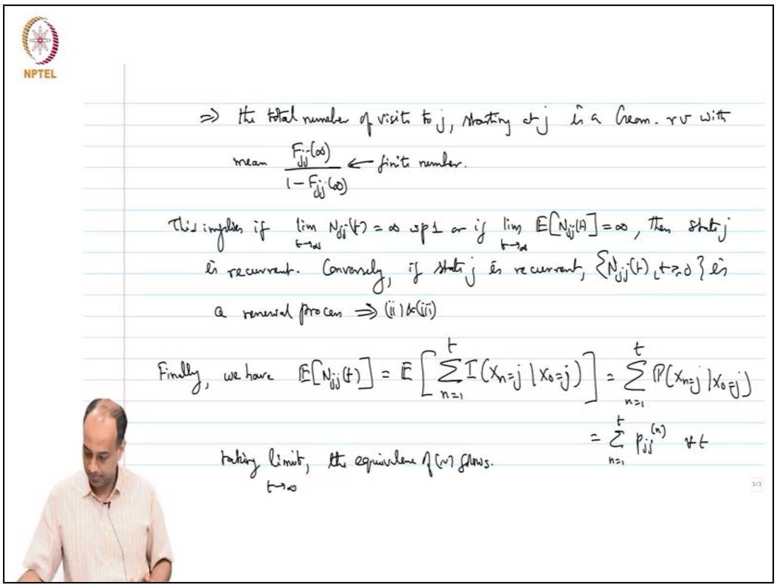

So, this implies or if limit t tending to infinity expected N jj t is equal to infinity then state j is recurrent conversely if state j is recurrent N jj t is a renewal process. So, that implies 2 and 3. So, we have proven the equivalence between 2, 3 and 1. Finally we have expected so, this is for proving 4 expected N jj t is the expected sum over indicators if I can put N equals point to t indicator X n equals j given X naught equals j.

Now this is a finite sum. So, I can take this inside that is equal to Y. So, why is this relation valid. So, N jj t simply counts the total number of times you have visited state j or state j until time t. So, you are counting simply the indicators X n is equal to j given X naught equal to j and something from n equals 1 to T you are looking at your summing all the indicators from n equals one to t whether X n equals j.

So, this becomes of course I can take the indicator inside because this is all a finite sum. So, I can take this becomes probability X n equals j given X naught equals j is equal to P jj n this is true for all t. Now therefore taking limits the equivalence of 4 follows. So, this n t taking limit t tending to infinity I can get the equivalence of the 4th statement with the other 3.

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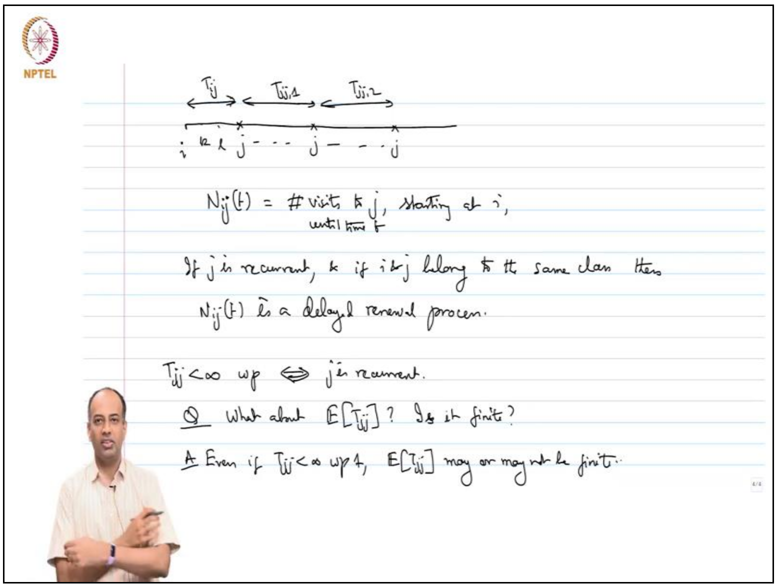

So, we have. So, far looked at starting a j and returning to j. Now what if I start at some state i then I look at subsequent returns to j ok this may be k, l whatever. So, if I start at i and look at the instances when i return to j what happens is that so, this will be a let us say some a typical time T ij and then subsequently again by strong Markov property you can argue that these will all be like your usual recurrence times of the state j starting at j.

So, if i say N ij lets say if I define this N ij of t is equal to total number of total number of visits to j starting at i if j is recurrent and if i and j belong to the same class then N ij of t is a delayed renewal process. So, the only difference now is that I am starting at some other state I which is in the same class as j. So, if y and j are not in the same class you may never go from i to j. So, you go from i to j in some finite time T ij and then you look at subsequent returns to j.

So, this T ij alone is atypical then you have typical recurrence times of state j to state j this N ij of t which counts the number of times you visit j until time t. So, number of with this to j until time t starting at i. This N ij of t turns out to be a delayed renewal process. So, what we know about renewal processes and delayed renewal processes including Strong law, Element Renewal theorem, Renewal Reward theorem for both renewable processes and delayed linear processes can be used to derive useful results about Markov chains.

Now we have seen that T jj less than infinity with probability one the same as saying j is recurrent. So, T jj the time between successive returns to state j if it is finite with probability one then you say that j is the current. Now what about the expectation of T jj. So, we know that T jj. So, then this returns to j will be a renewal process but what about the expected renewal time question is it finite is.

So, whenever you have a renewal process let us say iid X renewal process know you can always apply these renewal theorems. Now but you usually you have 1 over X bar you know coming in the denominator and all that one over expected X. So, we have to look at what is this expected? Expected value of this annual interval is it finite, see generally this expected T jj does not have to be finite ok even if T jj is finite with probability one. So, this we know this is basic probability even if T jj is finite with probability one.

Expected T jj may or may not be finite. So, it is possible to have a random variable T jj which is finite probability one by definition but its expectation may be infinity. So, this further subdivides these recurrent states into 2 types.

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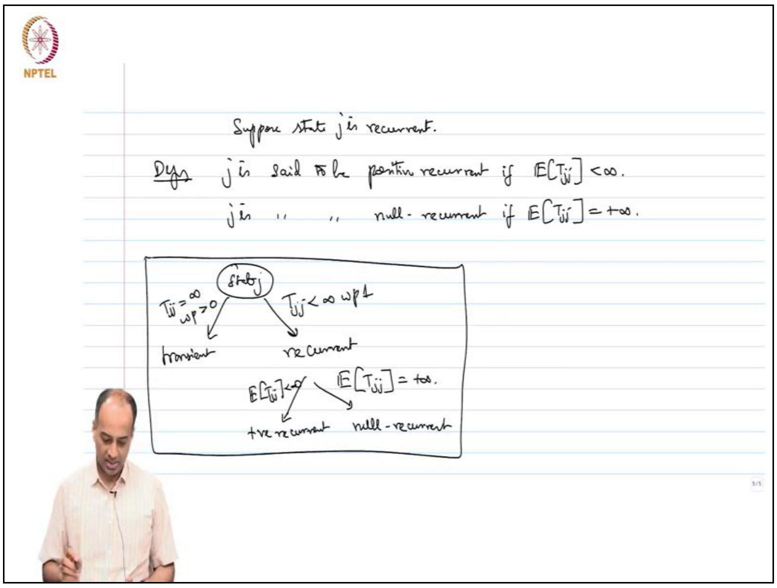

Definition, suppose state j is recurrent. So, suppose stage is the current definition j is said to be positive recurrent if expected T jj is finite j is said to be null recurrent if expected T jj equals infinity. So, this further divides states in a countable state DTMC. So, T jj is finite with probability one stage j is the current. If furthermore if T jj has finite expectation then the state j is said to be positive recurrent if expected T jj is infinity then we say that state j is null recurrent.

So, you could have a state. So, if you have a state j it could either be transient or current. So, if this is T jj is less than infinity with probability one you have a recurrent if T jj is equal to infinity with probability greater than zero. Then you have a transient state the current states are further subdivided into positive recurrent and null recurrent. Positive recurrent is if expected T jj is finite and a recurrent state is said to be null recurrent.

If expected T jj is infinity. So, that is the picture. now as it happens in the finite state Markov chains you never have null recurrence you have only recurrence and transients this null recurrence is something that occurs only in this countably infinite state Markov objects as you can show.

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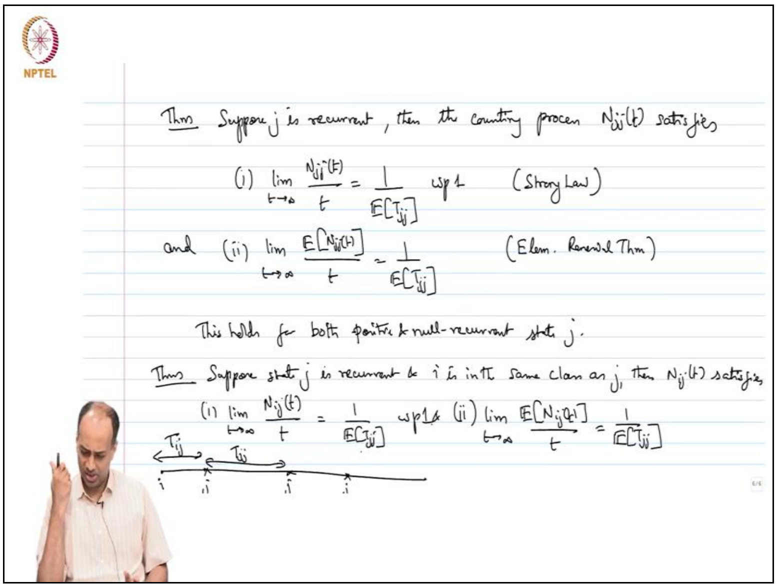

So, now let us start applying renewal theory and delay renewal theory to these return times. So, these processes suppose j is recurrent then the counting process N jj t remember N jj t is the total number of returns to j starting at j until time T this satisfies these 2 things limit t tending to infinity N jj t over t is equal to 1 over expected T jj with probability 1 and to limit t tending to infinity expected N jj t the expected number of returns to state j starting at j until time t is equal to the same thing.

So, this is nothing but Strong Law. So, N jj t is a renewal process. So, it must satisfy the strong law for renewal process and this is nothing but the elementary renewal theorem which you are familiar with all and this is true for both positive recurrent positive and null recurrent state j. So, remember that these strong laws for renewal process and element regional theorem are valid even if expected X is infinite.

So, we do not require that in this case we do not require that state j be positive recurrent it is ok if it is positive or current it is ok if it is null recurrent. And this does not require a separate proof because we have already studied strong law of renewal process in elementary renewal theorem. Similarly if you look at the N ij process, suppose state course state j is recurrent and i is in the same class as j then the counting process N ij t satisfies limit t tending to infinity N ij of t over t is equal to one over expected T jj with probability one and limit t tending to infinity expected N ij t over t equal to one over expected T jj.

Remember this I am looking at N ij t but here I have expected T jj not T ij. So, that is because if you if you remember the picture you are starting at i which is the same class as j but j is the recurrent state and for this I am looking at the number of returns number of visits to j until time T starting at state i. So, what really matters is this expected T jj this T ij does not matter if you remember from renewal theory we know that this first atypical time T ij does not matter.

So, limit T tending to infinity N ij t by t and its expectation are both equal to 1 over expected T jj. This is from this is basically strong law for delay renewal process and elementary renewal theorem for delayed renewal process. So, these are these results are quite useful.

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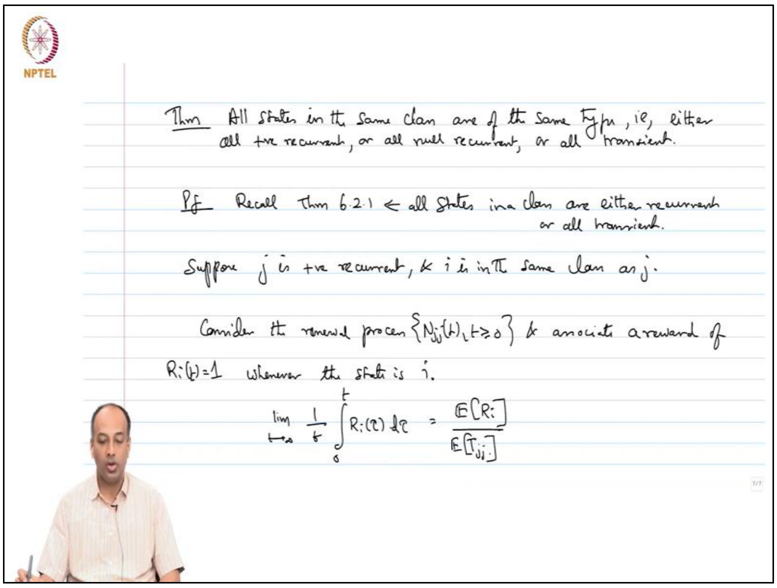

Now let us state an important theorem that says that all states in the same class are of the same type ie either all positive recurrent or all another current or all transit proof, See we remember that we have already proven that the states in the same class are all either current or all transient that we have already proven. So, now we have to we are just defining this if states in the same class are all or all either all positive recurrent or all the current are all transient.

So, we just have to prove the positive recurrent in all the current part. So, recall we already proved this theorem 6.2.1 in Gallagher this all states in a class are either or all transient. Suppose j is positive recurrent and i is in the same class as j. So, now I am looking at a class, so, j, i know is positive recurrent I am looking at some other state i in the class I want to prove that i is also positive recurrent I know that i is recurrent.

But I do not know whether it is positive or current that is what I have to prove consider the renewal process N jj t and associate reward of one let me say Ri of T equals 1 whenever the state is i. So, I am associating the reward process. So, I have a renewal process N jj t I know that j is positive recurrent and I am associating a reward process Ri of t which is equal to 1 whenever the chain is in state i.

Now I can apply the renewal reward theorem all which says that limit t tending to infinity 1 over t integral 0 to t Ri of tau d tau is equal to expected Ri over expected T jj.

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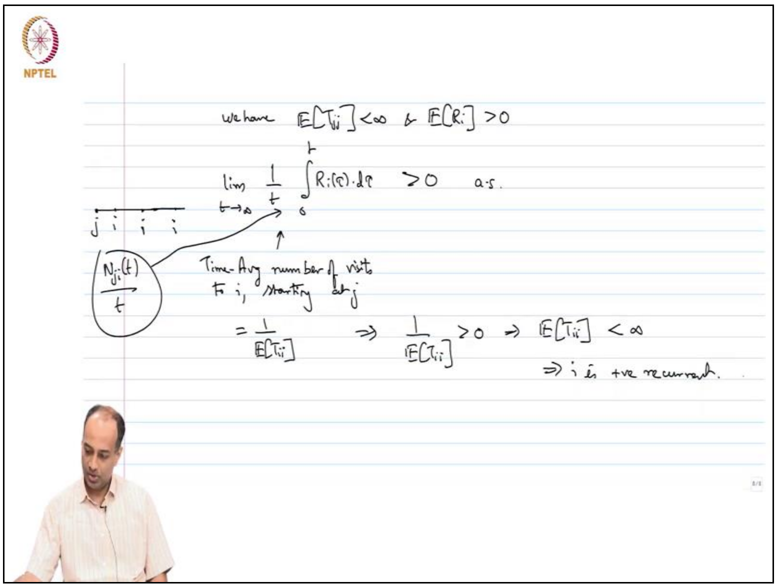

Now, this is nice this is from what? This is from renewal reward theorem. Now why is this relationship nice. Now we know that expected T jj is finite. So, we have expected T jj is finite because j is positive recurrent and expected Ri should be strictly greater than 0 why because i and j communicates they are the same class. So, whenever I am starting at j there is at least a positive probability that I will visit i.

So, the expected reward I get a reward of one whenever I visit i. So, the expected reward cannot be zero it has to be strictly positive because i and j are the same class. Therefore the RHS is strictly positive. So, this is of course almost surely almost surely. So, what does this mean limit t tending to infinity 1 over t integral zero to t Ri of tau d tau is something strictly positive divided by something strictly finite.

So, this would be strictly positive almost surely. Now what is this? This is the. So, remember you accumulate a reward of one whenever you are in state i. So, this is the number of times you visit. So, that this is the expected number of visits to state i starting at j this is nothing but is not the expected. So, this is the time average number of visits to i starting at j what is the time average number of visits to i starting at j? It is nothing but this is equal to 1 over expected T ii.

So, if you start at I sorry start at j and you are just counting the number of with this to i you are simply counting. So, this is just the N ji of t divided by t. So, that is what this, this guy is equal to that guy and the limit t tending infinity of this by you know the previous delayed renewal theorem by the delay removal theorem that we had in this here by this result this result here is equal to expected 1 over expected T ii.

So, that is excellent. So, this implies one over expected T ii is strictly positive implies expected T ii is less than infinity implies state i is positive recurrent. So, if I start with the positive recurrent state j I have shown that any other state i in the same class must be positive recurrent. So, even if there is one positive recurrent state in the class then the entire class is positive or current. So, this proves what I want.

So, I am done this proof is done but this equation I want to go back to this equation which is you know I want to rewrite this maybe in words. So, let me call this star. The star will be useful later.

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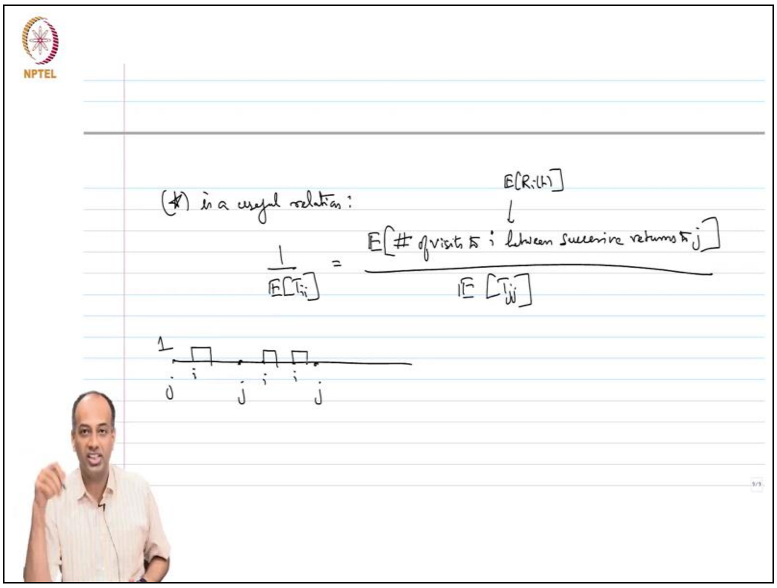

So, what does it say? It says one over expected T ii which is the left hand side equals expected number of visits to i between successive returns to j returns to j divided by expected T jj. And this relationship is valid as long as j is recurrent and i is any other state in the class that is all that I need I do not even need positive recurrence for this to be true this is just this is almost surely sorry. So, these are just numbers there is no almost surely here.

So, left side left side of that integral is equal to one over expected T i bar by a strong law ok almost surely and is equal to expected number of visits. So, this is nothing but expected Ri t. So looking at this process starting at j and you accumulate a reward of whenever you are in state i your accumulator reward of 1. So, expected Ri t is the expected number of visits to i between successive visits to state j divided by expected T jj.

So, this is a very useful relationship, we will probably come back to this or you can rearrange and write it as expected number of visits to i between successive returns to j is equal to expected T jj by expected T ii. This will be useful when we do steady state probabilities and all that; this module can be stopped here.