Stochastic Modeling and the Theory of Queues
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Lecture –56
The Strong Markov Property

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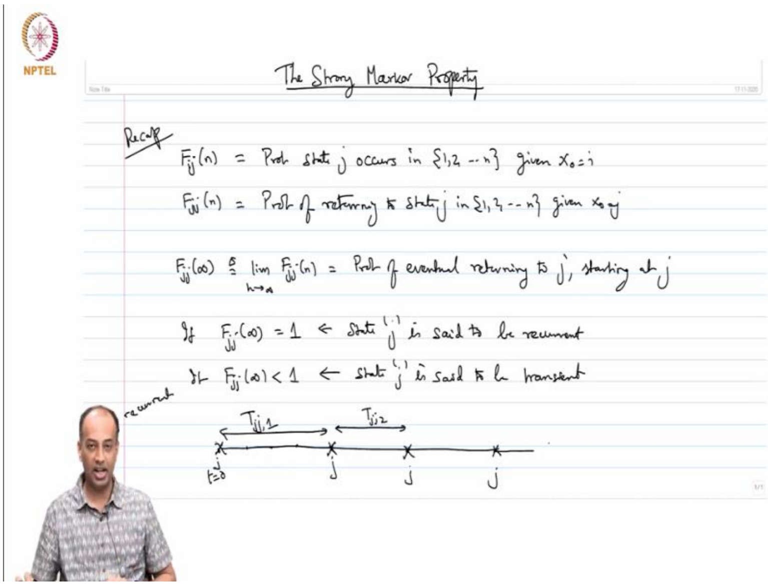

Welcome back. In the previous module we were looking at first passage times and recurrence times. So, if we recall we defined let us say for states i and j we defined F ij of n as the probability that state j occurs in time 1 through n given we started at state i and similarly F jj n is the probability of returning to state j in time 1 through n given we started in state j. Then we said these F ij n and F jj n are monotonically increasing in n.

Therefore as n tends to infinity this limit has to exist. So, F jj infinity is the probability this limit has to exist as n tends to infinity Let me just say is defined as limit n tends to infinity F jj n this limit exits since F jj n is monotonic this is simply the probability of eventually returning to j starting at j. Now we said that if the probability of eventually returning to j starting at j if this probability is equal to 1 then we said the state j is recurrent.

We said if F jj infinity is equal to 1 then say state j is said to be recurrent and if F jj infinity strictly less than 1 then we say so in this case if F jj infinity is strictly less than 1 then there is a positive probability of never returning to j having started at j. In this case state j is said to be transient. So, all of this is just recap from last time. Now, suppose j is a recurrent state let us say at time t = 0.

I start at state j then I go through a bunch of transitions and I hit j again at some point. So, if j is recurrent I will keep coming back to j that is the whole understanding of recurrence. So, what we sort of intuitively argued is that so let us say j is recurrent so let us say this T jj is my first let us call it T jj 1 to be precise T jj 1 is the first time I come back to j this is the first recurrence time of a state j.

Now having come back to state j at this point we intuitively argued that the further evaluation of the Markov chain here should be statistically identical and independent of whatever happened in this T jj 1 interval. So, we intuitively argued that this T jj 2 the second recurrence time of state j must be independent because of Markov property of the past and should also be identically distributed to T jj 1 because the Markov chain is homogenous.

So, this let us believe that these instances if you are starting at j the times at which you hit j constitute a renewal process. This is what we intuitively figured out. Now for a recurrent state j it turns out to be true and this renewal process of these return times to j will be very useful in studying the long term behaviour of countable state Markov chains. However, this intuitive understanding is correct that these T jj constitute iid random variables, but to prove it rigorously takes a little bit of work I will tell you why this is the case?

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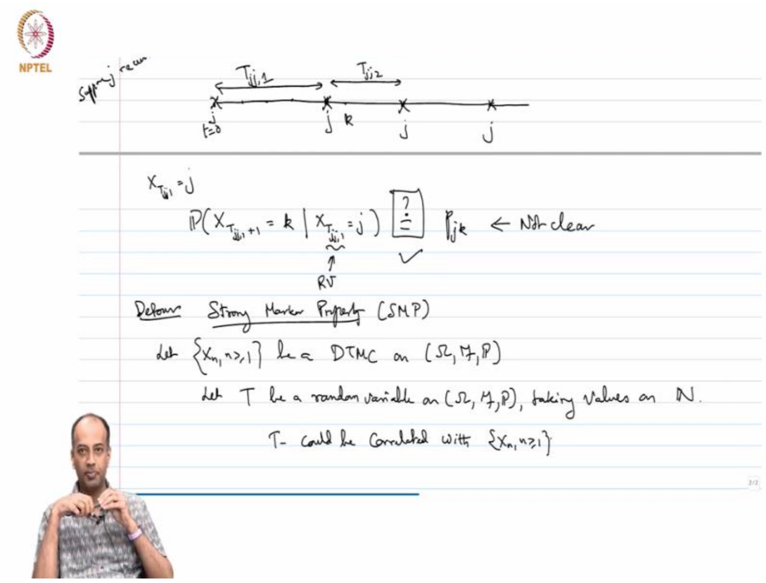

So, if you look at this time T jj 1 the first time of coming to state j it is a random variable. So, this T jj 1 is not a deterministic number. So, for example, T jj 1 so if I look at probability that X so X is of the random variables. So, I know that X T jj 1 is just j. Now I want to argue that the further evaluation of the Markov chain is statistically identical to the Markov chain that started at time 0.

So, what I want to do is look at T jj 1 + 1. What is the probability that this is some state k given X T jj 1 equals j. Is this equal to question mark this is a big question mark. Is this equal to P jk? See at T jj 1 I am at state j I have hit j for the first time after I have come back to j. Now I want to look at the further evaluation of the Markov chain in this what I want to prove is the second renewal interval.

So, what is the probability that I go to some state k here. So, what is the probability X T jj 1 + 1 = k given X T jj 1 = j. Is this equal to P jk? Intuitively, it seems that this should be true. However, please note that this guy T jj 1 is not a number this is a random variable and that itself depends on the previous values of the states of the Markov chain. So, T jj 1 is random variable.

So, what you are looking at is you are looking at some sort of a transition probability from j to k where the index is no longer deterministic n. This is not like T X n + 1 equals k given X n equals j. Here the n itself is T jj which is a random variable. So, it is not clear at all at least it requires a proof that this is the case. It turns out that this is true. It turns out that this is correct, but not clear.

This is not clear immediately and the reason that this is true is a non-trivial matter and this property is known as the strong Markov property and that is the subject of today's discussion. This you can view as a little bit of detour. So, we will use this detour to learn strong Markov property which is an important property and using strong Markov property we will prove that this is equal to P jk the way we wanted.

And then our renewal theory will be useable. We can rigorously argue T jj 1, T jj 2 etcetera are iid and therefore we have a renewal process. So, what this strong Markov property? I have to define this properly. So, this is the detour into strong Markov property or SMP. So, let us say these X n let X n, n greater than 1 equal to or (()) (09:40) be a DTMC on some probability space on omega, F, P..

So, all these random variables X n come from they have to come from probability space let me call as omega F, P. So far I am not really bothered about what this omega F, P is, but let the reason I am doing this is I am going to consider let T be a random variable on omega, F, P taking values. So, it is a natural number valued random variable. Let us say taking values on N.

Now this T so T could be correlated in some way with these X n which is the Markov chain sequence. So, what we are given is some probability space omega, F, P on which this Markov sequence is DTMC X n sequence is defined. I am going to consider some other random variable T which is a time random variable taking values in N and this T could be something correlated.

It may have something to do with the Markov chain. So, T takes values in 1, 2, 3 dot, dot, dot it is a natural number valued random variable defined on the same probability space and it could have something to do with the process X n that we are considering. For example this T could be the first time you hit j or something like that. There is a first time the process X n goes to j.

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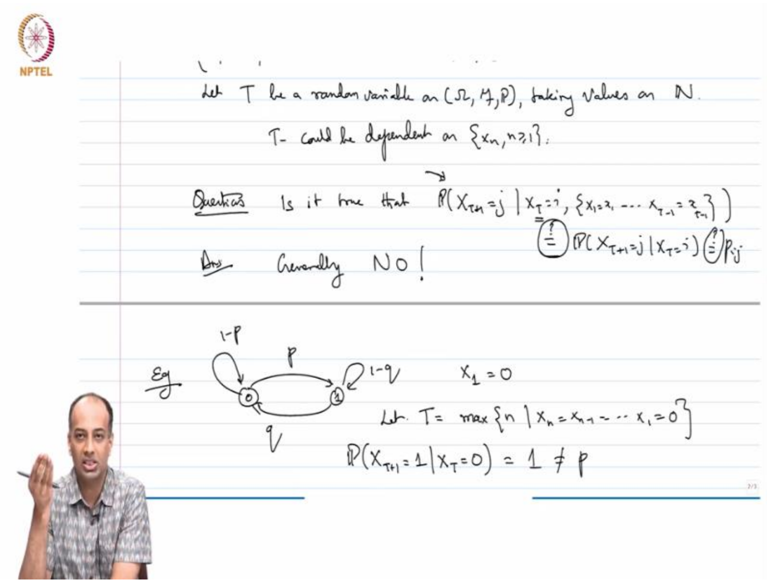

The question is it true that probability that X if you look at X T + 1 = j given X T = i = P ij. So, am I saying I am looking at some random variable T which could depend on the process

X. I am looking at that random T given that at random instance T X T = i what is the probability that X T + 1 equals j. Is it equal to P ij? If this T is some deterministic N then I know that this probability X n + 1 equals j given X n equals i is simply P ij because the Markov property holds.

Is it true also for some random variable T which could depend on the process X. This is the question we want to answer. The answer generally is no. Actually I can ask this even further so let me define this question a bit more. Is it true that even Markov property holds? So, maybe I should ask is probability of so let me just look at this X T = i and then X let me say X 1 = x 1 dot, dot, dot, X T – 1 equals X T – 1 so you fix whatever values for states between 1 through T – 1 and X T = i.

Is this equal to probability X T + 1 X T + 1 equals j given X T equals i. Is this true and is this equal to P ij. So, this equality which is I am not saying this is true. In fact, I am saying that this not true in general. So, this particular equality here is asking whether the Markov property is true at some random time T. So, what you have here is this big probability is the probability of X T + 1 equals j given the entire history.

Is it equal to the probability that X T + 1 j equal to j given just X T = i is it true? Answer is generally no and even if that is true is it equal to the usual transition probability from i to j. It is not at all clear. In fact the answer is generally no. So, I can show you an example to show that this is not the case which I will do soon. The strong Markov property is the property that if T is a random variable like defined and if this property holds.

Then the random variable T is said to satisfy the strong Markov property. So, let me just show you an example. Let us take this Markov chain 0, 1 just a two state Markov chain this is P and this is q 1 – q 1 – P this is a very simple Markov chain. Let me define let us say that I started X 1 = 0 I am starting at 0 and you look at T as the largest N such that X n = X n – 1 = dot, dot, dot X 1 = 0.

So, I am starting at 0 for this Markov chain simple two state Markov chain and I am defining T to be the random instant n for which the largest n for which the first nth length run = 0. So, this is my definition of T. So, what does T mean? I want to have so if T = 13 that means that

X 1 through X 13 are all equal to 0 and X 14 must be equal to 1 then T = 13 that is what this definition means let T equal to.

Now for this T can you say for example can you look at probability that what is probability that X T + 1 equals 1 given X T = 0 what is this equal to? This is equal to 1 this is by the definition of T. This is not equal to going to the probability of going from 0 to 1 and in fact you can argue that. So, this is just showing a second part. So, for this random variable T the probability of X T + 1 equals 1 given X T = 0 is not your original Markov chain transition probability.

You can also show that this sort of the first equality also does not hold. It does not have this Markov property at all. So, generally these equalities are not valid. So, bottom line is that if T is a random variable which could be correlated which could dependent may be instead of correlated let me just say could be dependent on this X n for n greater than or equal to 1. So, if I have some random variable T which could depend on these process X n then if I look at whether the Markov property holds for at index T generally the answer is no.

If it happens to hold then the random variable T is said to satisfy strong Markov property for this DTMC.

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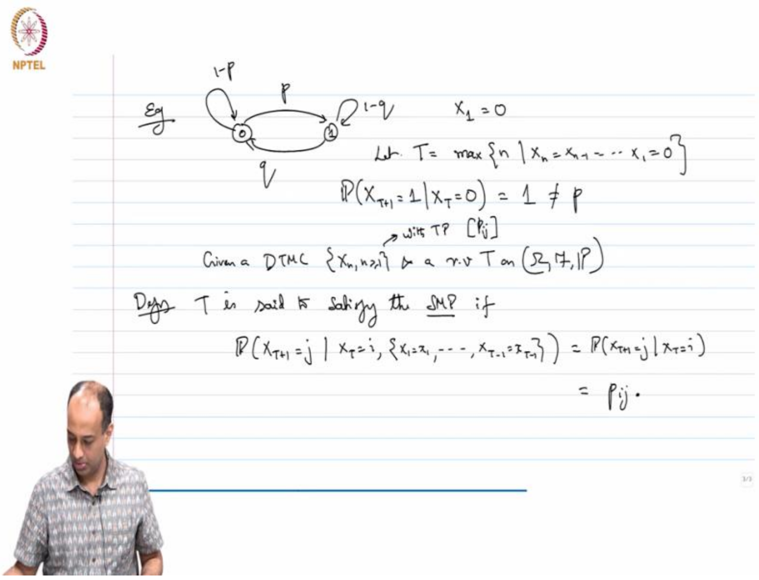

Definition so given a DTMC and a random variable T on the same probability space definition T is said to satisfy the strong Markov property if probability that X T + 1 equals j given X T = i and whole bunch of other history that is like X 1 = X 1 dot, dot, dot X T – 1 =

some X T – 1 equals P X T + 1 equals j. Given X T equals i equals P ij. So, the DTMC I should say given a DTMC with transition probability matrix P ij.

So, I am given a DTMC with a transition probability matrix P ij does not have a matrix it is just a transition probabilities P ij could be countably infinite state Markov chains. So, maybe I should just say transition probabilities P ij I want the probability that X T + 1 equals j given X T equals i and whole bunch of history should not depend on the other rest of the history. It should only depend on X T equal to i.

And it should be equal to my original transition probability matrix. I am not saying this is always true if this is true than the random variable T the random time T is said to satisfy the strong Markov property. Clearly, in the previous example I showed that random variable T defined above does not satisfy the strong Markov property. So, it is special property it does not always hold.

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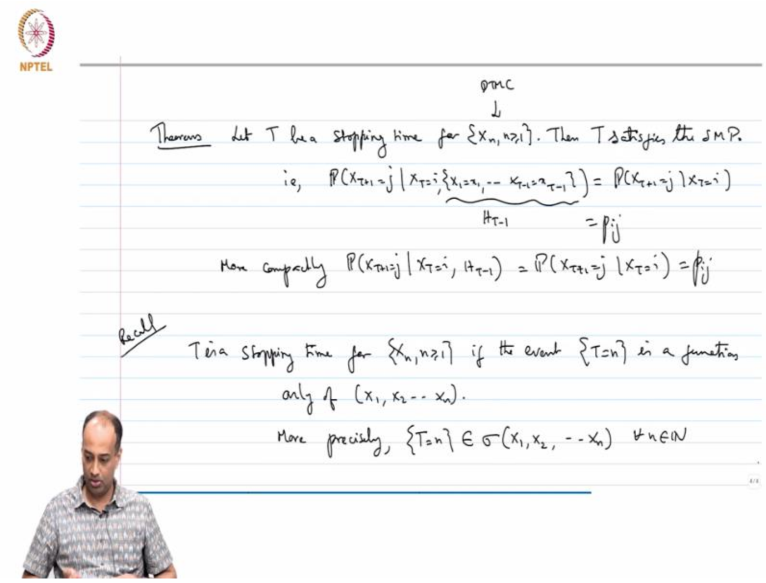

So, the big theorem about Markov chains is the following. Theorem let T be a stopping time so recall what a stopping time is so we studied stopping time earlier T be a stopping time for this DTMC. The sequence then T satisfies the strong Markov property. So, T should be a stopping time it should be; it cannot be defective stopping time it has to be finite with probability 1.

So, if T is a stopping time for the Markov chain X n, n greater than or equal to 1 this is your DTMC then T satisfies the strong Markov property ie probability that X T + 1 equals j given

X T equals 1 and whole bunch of history equals probability X T + 1 equals j given X T equals i equals P ij. This is true for all I mean I should have said this for all i, j in S and all possible values of X 1, X 2 dot, dot this entire history.

These could be any states in the Markov chain it does not matter what this history is. So, more compactly I can write this as if I just denote this as some history at time T – 1. So, more compactly I can write this as I want to show that probability that X given X T equals i X T – 1 is equal to probability that X T + 1 equals j given X T equals i should be equal to your original P ij.

So, just to refresh what is a stopping time T is the stopping time for X 1 for X n means that recall T is a stopping time for X n if the event T equals n if a function only of X 1, X 2, dot, dot, dot X n. So, looking at my first n random variables I should be able to decide whether I stop at n or not. More precisely this T even T equal to n must be measurable under the sigma algebra generated by the random variable (()) (27:02).

So, X 1 through X n they generate a sigma algebra and the event T equal to n must be in the sigma algebra for all n and N. So, this is if you are used to the sigma algebra language this is it may not be easier to understand, but this is the precise definition. But for all practical purposes event T = n is determined completely by looking at the first n random variable and if that is the case. If T is a stopping time for discrete time Markov chain then the Markov property holds.

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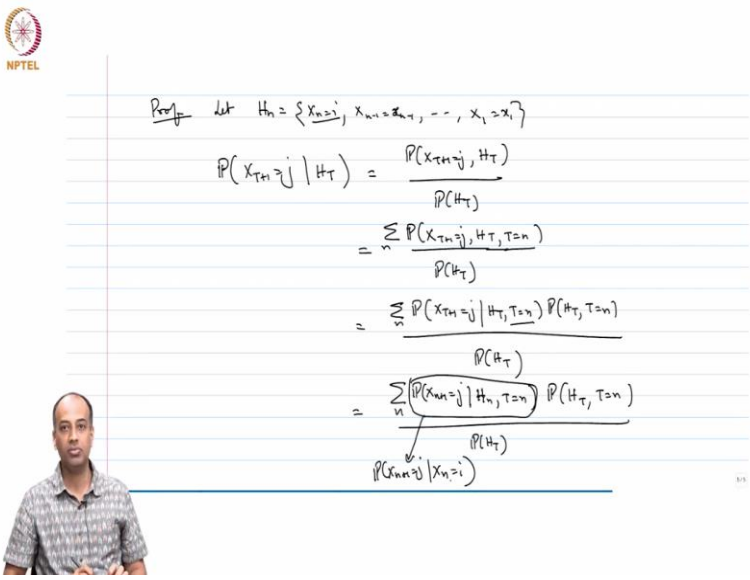

Now we have to prove this proof. So, I am going to write let H n (()) (28:06) at time n is defined as I am taking X n = i and then X n – 1 = little x n – 1 dot, dot, dot X 1 = little x 1. So, I am looking at a history at n. So, we are looking at probability that X T + 1 equals j given H T. So, in H T I have included this so it is because of this so H T is equal to i.

So, I am taking my conditioning on this H T I am looking at X T = i and a whole bunch of history. This is equal to so I want to prove that this is equal to probability of X T + 1 equals j so I want to prove that this is equal to P ij. So, that requires some proving. So, this is how it goes. So, this is equal to probability that X T + 1 equals j intersection H T over P of H T. Now, I am going to use total probability so I am going to write the denominator is the same.

P H T I am look at probability X T + 1 equals j H T T equals n and sum over n sum over all natural numbers n. Now why this is valid? This is by the law of total probability I am just using total because T equals n is these are disjoint events if you stop at 13 you will not stopping at 14 or 12. Now this is equal to sum over n probability X T + 1 equals j given H T, T equals n.

Probability H T, T = n over probability H T this is just be conditioning definition of conditioning. So, I have conditioned on T equal to n so this I have done that I can write this as sum over n probability X n + 1 equals j given H n now T = n probability of H T and intersection T equals n over probability of H T. Now is the crucial step if you look at this guy so this is condition on H n.

Remember that H n I have X n = i so if I am looking at a probability that X n + 1 equals j given H n and T equal to n. Remember T = n is completely a function of X 1 through X n because of the stopping rule property T equals n is completely determined by X 1 through X n. So, I am conditioning on the event H n which includes all the values from X 1 through X n and in particular X n equals i.

And T = n is also completely determined by these random variables. So, bottom line is that this circled event this can be showed to be circle probability this can be showed to be equal to probability that X n + 1 equals j given X n is equal to i that is because T = n is completely determined by this H n the history H n.

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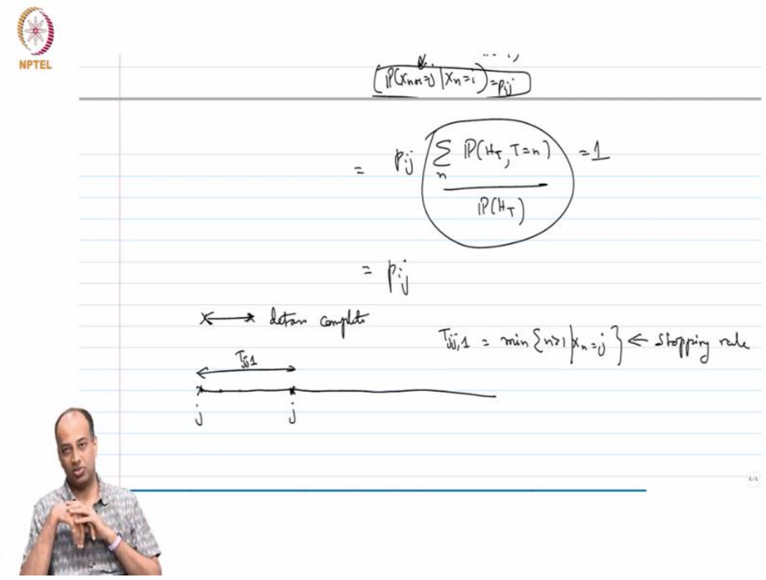

So, this is a crucial step you may want to explicitly write this out. So, that just comes out this guy is nothing, but P ij. So, this just comes out of the summation P ij times summation n probability H T, T equals n over probability H T. H T, T is equal to n. Now this is equal to 1 why? Because I mean this is just total probability. So, this is equal to P ij as we wanted. So, the key step is in realizing this that this circled probability is equal to just T X n + 1 equals j given X n equal to i which is equal to P ij.

And this is because T equals n is determined completely by the history H n and in H n we have X n = i is the only thing that matters. So, we can use Markov property the previous history in H n does not matter for fixed n. So, this proves the strong Markov property for stopping rule for Markov chain. So, this is great we have taken these detour approved strong Markov property which says that the Markov property holds for stopping rule stopping times defined from the Markov chain.

So, detour over this completes the detour. Now, if you are back to this renewal view of the world you look at your back to T jj and F jj now. So, you are starting at j and then you are going to a bunch of states and you are coming back to j. This is my T jj 1 now you can show that T jj 1 which is the minimum of those n for which X n = j minimum n greater than so I am starting at minimum n greater than 1 I should say I think minimum n greater than 1 for which X n is equal to j.

This is a stopping rule why? Because I can determine whether T jj 1 = n by looking at the first n values of the states in the Markov chain. So, I can look at X 1 through X n and decide whether or not T jj 1 is equal to n meaning that I should have come to j for the first time starting at j. So, this can be clearly shown to be a stopping rule which means that from T jj 1 onwards the strong Markov property holds.

Why T jj 1 is stopping time for Markov chain and by the strong Markov property the further evaluation of this Markov chain is the same as starting at time 0. Although, this T jj 1 is a random variable it is a stopping time for this Markov chain. So, the statistical evaluation is the same as the original Markov chain. If the T jj 1 were not a stopping time if it were some other crazy random variable like I defined earlier.

Then this may not hold at all I should you an explicit example where this is not equal to P ij. So, if you are going to look at let me write that down so if you look at this guy if you go back and look at this I initially said this is not clear, but now it is clear that this is true because T jj 1 is a stopping time. In fact, you can now write probability of T jj 1 + 1 equal k T jj 1 + 2 is equal to l given X T jj 1 = k is equal to the further evaluation will be Markovian still. The statistics do not change that is the bottom line.

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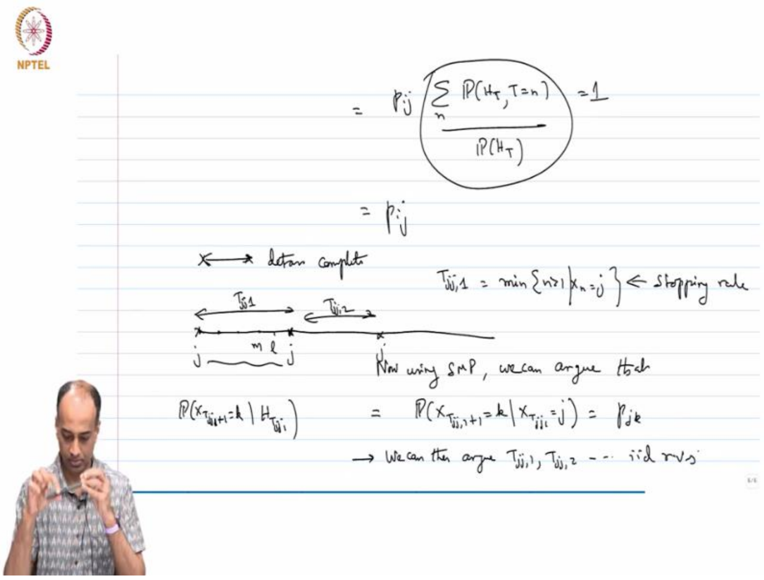

So now we can argue strong Markov property we can argue that let us say P we can argue that P X T jj 1 + 1 equals K given X T jj 1 equals j = P jk. Not only that it does not matter how do I got to this is true it does not matter however I got to these state j does not matter. So, I can in fact just say that probability that X T jj 1 + 1 equals k given all the history H T jj 1 whatever it is.

This entire history does not matter is equal to all that matters is that you will end in J and then this will be equal to P jk and now the Markov chain evolves like it originally did. It is as though once you come back to j it is as though T = 1 all over again or T = 0 or wherever you start. So, now we are justifying in saying that the further this evolution this T jj 2 is identically distributed because the transition curves are exactly the same.

And it is independent of due to the strong Markov property the independent of the previous states that it may succeed in getting to j. So, now we can apply then argue T jj 1 T jj 2 etcetera are iid random variable if j is recurrent and then there will be renewal every time we will turn to j will be a renewal instant and then you can happily apply all the tools we know from renewal theory key renewal theorem, renewal reward theorem everything can be applied.

Strong law for renewal process is elementary renewal theorem all of these will apply if as long as state j is recurrent. So, this module I spent in proving this property strong Markov property because it is anyway quite useful and in particular this intuitive understanding of this recurrence to state j being a renewal process can be formalized using a strong Markov property. Thank you. I will stop here.