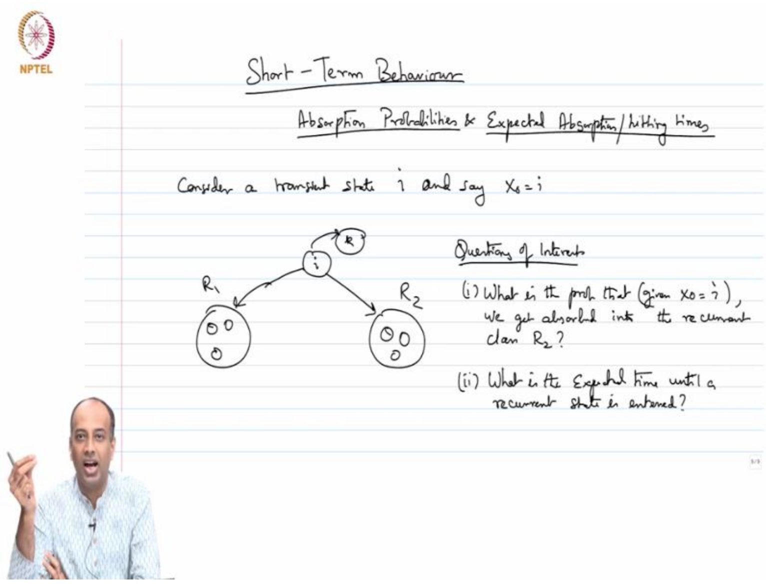
Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology – Madras

Lecture –53
The Short-term Behaviour of a DTMC

(Refer Slide Time: 00:15)



Welcome back. In this module we will discuss the short term behaviour of DTMCs. So, far we have studied long term behaviour that is if you start off a Markov chain in some state and connect for a very long time does it converge it to a stationary distribution or not after a long time that is a different consideration from what we are going to consider now which is short term or transient behaviour.

So, the question here are like if you start at a transient state how long would it be before you get absorb into a recurrent class. For example, if there are multiple recurrent classes what is the probability that you get absorbed into one of these recurrent classes. So, these are the kind of questions that we would wish to answer in this particular module. So, let us consider a transient state i.

So, this is the starting state so you say that and say X 0 equals i. So, you are starting at a transient state i let us say you are starting at some transient state i at time equal to 0. There are many states and let us say there are some this is one recurrent class R 1 this is some recurrent class R 2 with a whole bunch of states inside them. So, there will be some transitions inside the transient class.

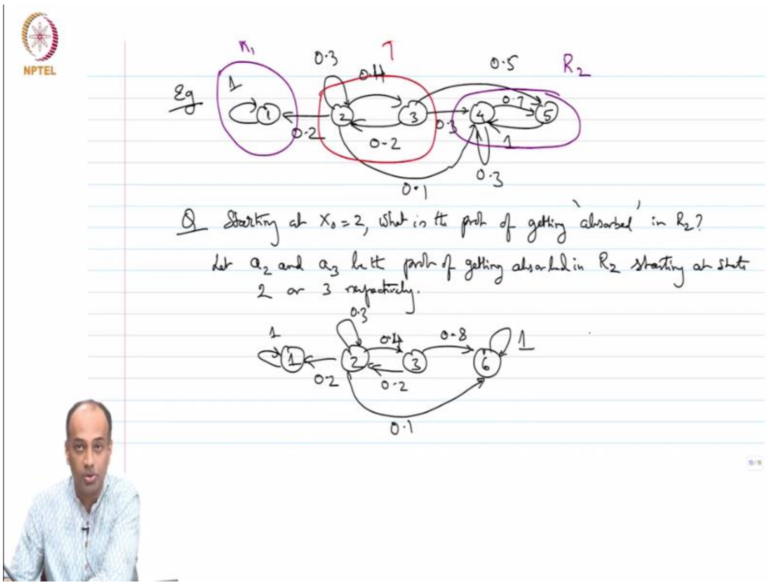
And there maybe some into the recurrent class 1 some maybe into the recurrent class 2 and so on. Now starting at i the questions of interest will be as follows. What is the probability that starting at this transient states i we get absorbed into the recurrent class R 2 for example or R 1 does not matter. So, you could have a number of transient classes and number of recurrent classes and let us say you are starting at one of those transient states.

And I want to look at the probability of getting absorbed in a particular recurrent class. Of course, with probability 1 if you start at a transient state you are guaranteed that with probability 1 you will enter one of the recurrent classes, but you do not know which one you will enter. You will enter R 1 on one turn of the Markov chain and then stay in R 1 and a different turn of Markov chain you could start at i and enter R 2 and stay in R 2.

So, what is the probability that you start in i and enter say R 2 the recurrent class R 2. This is a question of interest. Another question of interest is how long before you get capture. What is the expected time until a recurrent state is entered. These are the kind of questions we want to discuss this is what I mean by short term behaviour. The long behaviour we know that in geometrically fast we are going to get out of the transient state transient class.

But I want to precisely characterize with what probability do I go to which recurrent class and what is the expected time to get into that recurrent class. These are the kind of questions I want to talk about.

(Refer Slide Time: 04:39)



It is easiest to discuss these sort of questions using examples and then generalize it. So, I am just going to consider an example of this Markov chain. Let us say I have a 5 state Markov chain in which state 1 is absorbing. So, if you are in state 1 you are always in state 1 and in the other transition probabilities are as follows 0.2, 0.3, 0.1, 0.5, 0.3 and so (()) (05:28) this is 0.7 and that is 0.3 this is 0.2.

So, I have just taken an example. This is a 5 state DTMC with the transition probability is given. If you look at these states it is very clear immediately that state 1 is a recurrent class by itself and if you look at the state 4 and 5 once you are in state 4 and 5 you can go from 4 to 5 and you can go from 5 to 4, but you cannot get out from this 4, 5. Once you are in one of these 4 or 5 you will always remain in 4 or 5 which means that this 4 and 5 is another recurrent class R 2.

And if you look at this 2 and 3 you can go from 3 to 4 or 3 to 5, but you can never come back from 4 to 3 or 5 to 3. Similarly, you can go from 2 to 1, but you can never comeback from 1 to 2. So, that means that these states are transient 2 and 3 are transient. Now the question is starting at let us say starting at X 0 = 2 what is the probability of getting absorbed what is getting absorbed? Meaning you start at 2 and you reach this R 2 getting absorbed in R 2.

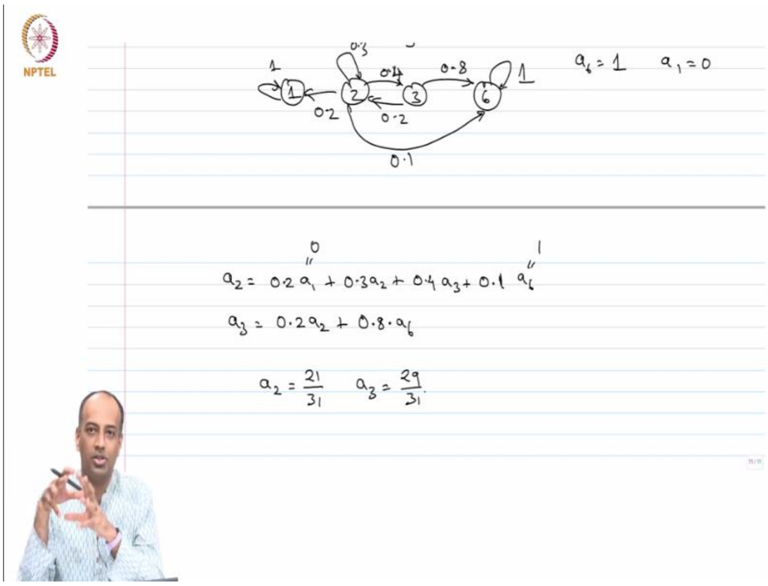
So, you get into one of these 4 or 5 and you remain there. So, you have to get captured either in R 2 or get absorbed in R 2 or R 1. I am asking starting at state 2 what is the probability that you get absorbed in these recurrent class R 2 or starting at state 3 for that matter. So, this can be calculated as follows. Let a 2 and a 3 probability of getting absorbed in R 2 starting at state 2 or 3 respectively.

So, now if you notice once I reach state 4 or state 5 in R 2 I do not care I am going to remain in state 4 or 5. So, if I start in state 2 and eventually end up in state 4 or state 5 I do not care. I am going to define in the different class. So, what I am going to do is to draw a modified Markov chain in which the recurrent class R 2 is converted to a single state which is absorbing.

So, I am going to keep this part of the Markov chain it is same so 1, 2, 3 everything here is the same all these probabilities are the same 0.1, 0.2 except I am going to call a nu state 6 which is an absorbing state from 3 what is the probability that I go to either 4 or 5 it is 0.8 it is 0.5 + 0.3. So, if I am at 3 the probability of going to one of these 4 or 5 is 0.8. Likewise, if I am in state 2 the probability of going to state 6 is 0.1.

And once I am in state 6 I am going to be in state 6 forever sorry I got this wrong this is 0.4 otherwise they would not add up to 1. Now, here I want to ask what is the probability that starting at state 2 I get absorbed in state 6. I have to get absorb in either 1 or state 6 what is the probability that I get absorbed in state 6. Now, this can be calculated using the following set of equations. This is by using the theorem of total probability.

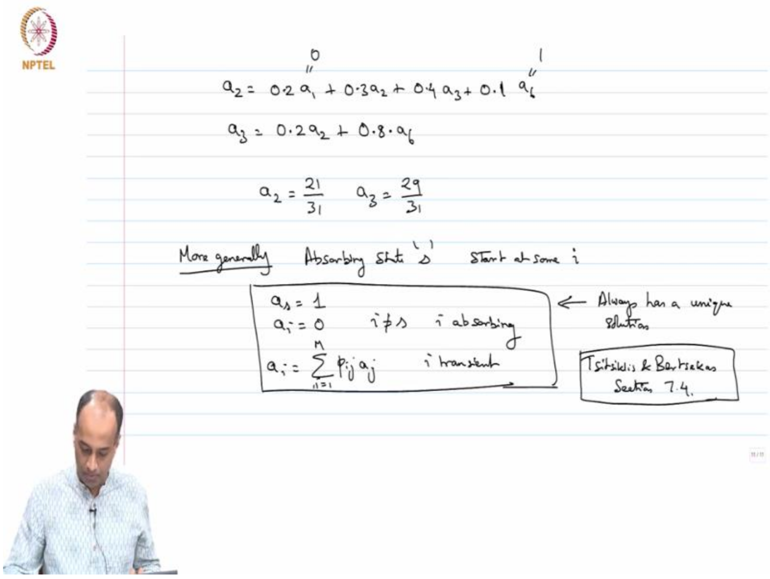
(Refer Slide Time: 10:23)



a 6 = 1 because if I start at 6 I am going to get absorbed at state 6 in probability 1 and a 1 if I start at state 1 I am not going to get absorbed in state 6. So, I will say a 1 = 0. Now, I can write a 2 = 0.2 times a 1 + 0.3 times a 2 + 0.4 times a 3 + 0.1 times a 6 and then for a 3 I am going to write a 3 the probability that I start at 3 and get captured at 6 eventually is equal to 0.2 times a 2 + 0.8 times a 6 that is it.

Now, if you solve so this a 6 is of course equal to 1 and a 1 = 0. If you just go ahead and solve this and you will get a 2 = 21 on 31 and a 3 = 29 on 31 so you get something. The bottom line is that you can write out these equations and calculate the probability that you get captured in this recurrent state R 2 which I have just collapsed into one absorbing states.

(Refer Slide Time: 12:08)

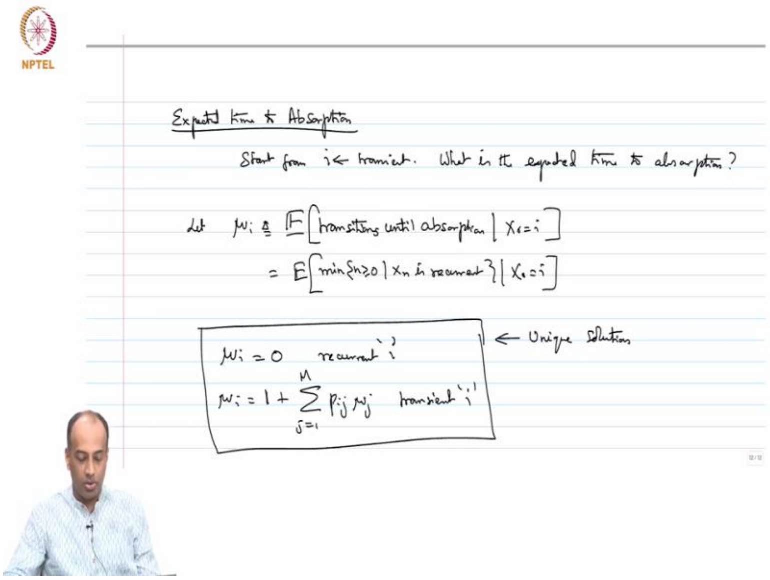


More generally I can write the following. Let us say we have an absorbing state let me just say there is one absorbing state there is an absorbing state S which could be a collapsed version of a recurrent class like I did above and start at some i then you can write out the following equation a s = 1. So, if you start at s you are going to get absorbed in s with probability 1 a 1 = 0 for i not equal to s i absorbing like we had above if a 1 was 0.

If you start at an absorbing state which is not your absorbing state s we are going to remain in that i. So, you are not going to get absorbed in s and then finally for all transient i you have the equation sum over j equals 1 to M P ij a j for i transient. So, these are the equations governing if you start at i what is the probability that you get absorbed into the particular absorbing state s?

This is the set of equations you have to solve. This is just by the theorem of total probability and it can be shown that this system of equations always has a unique solution. I am not going to prove uniqueness here. So, this general discussion in this lecture can be found in the book of Tsitsiklis and Bertsekas which is an undergraduate text on probability and this is section 7.4. So, this entire topic is covered in this particular book.

(Refer Slide Time: 14:53)



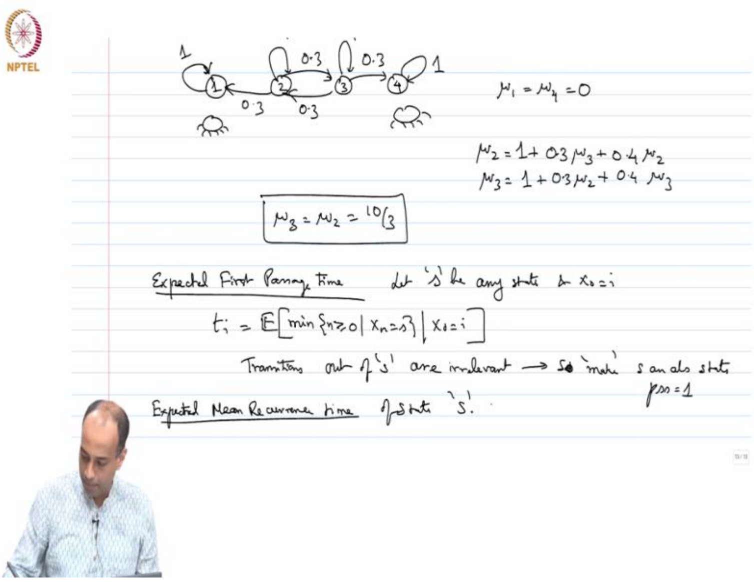
Next, we move on to talking about this absorption times expected time to absorption or expected or expected hitting times (()) (15:13) known. So, you start from i it is a transient state what is the expected time to absorption to a recurrent state. Let nu i be defined as expected number of transitions until absorption given X 0 equals i that is nothing, but expected recurrent.

Given X 0 equals i so these equations nu i can be shown to satisfy a different set of system of equations using the theorem of total expectations. In fact we can show that this mu i satisfy mu = 0 for all recurrent i and mu i = 1 + sum over j equals 1 to M P ij mu j for all transient i. This is the system of equations that one needs to solve. Intuitively, this mu i is the expected number of transitions until absorption.

This is the set of equations that we have to solve for mu i. Intuitively. mu i is the expected number of transitions until absorption into a recurrent class. So, mu i is 0 for all recurrent i and mu i = 1 + sum over P ij mu j. So, this interpretation of this equation is that you start at i you take a step to go to state j and from state j you have this expected time to absorption is mu j and the probability of going to j is of course P ij.

So, this is just the total theorem of total expectations. Now this also has a unique solution. This also can be shown I am not bothering to show it. This is the system of equations to solve for the expected time for absorption.

(Refer Slide Time: 18:44)



To show you an example from the book by Bertsekas and Tsitsiklis you consider this Markov chain the 4 state Markov chain. In state 1 and in state 4 there are 2 spiders and there is a fly that keeps jumping around between these states. So, from state 2 to state 3 so these are the jumping probabilities of the fly. So, the fly jumps forward or backward with probability 0.3 and stays in the same state with probability 0.4.

But if it jumps forward with probability 0.3 or from state 3 and to reach a state 4 then the spider will eat the fly. In that case the fly will not come back. Likewise, state 1 also if the fly jumps from state 2 to state 1 the fly will be eaten by the spider and so in that sense state 1 is absorbing. Suppose, so here let us say you are starting at state 2. Starting at state 2 what is the expected time until the fly gets eaten by the spider?

So, we have mu 1 = mu 4 = 0 because they are recurrent and for transient states 2 and 3 are transient states we have the equation mu 2 = 1 + we are starting at 2 and with probability 0.3 you are going to state 3. So, 0.3 mu 3 + 0.4 mu 2 and likewise mu 3 = 1 + 0.3 mu 2 + 0.4 mu 3 and if you solve the system of equations you will get mu 1 so mu 3 = mu 2 = 10 upon 3. So, whether the fly starts at state 2 or state 3 the expected number of transitions until the fly is eaten by spider is 10 over 3.

So, this is an example from the book by Bertsekas and Tsitsiklis. Now, this is basically you start a transient state what you have talked about is start at a transient state and you calculate the expected time to absorption. Now we can also talk about expected first passage times. Expected first passage time is the expected time that you take to reach some state i given that you started in so expected time until you reach a state s given that you have started in state i.

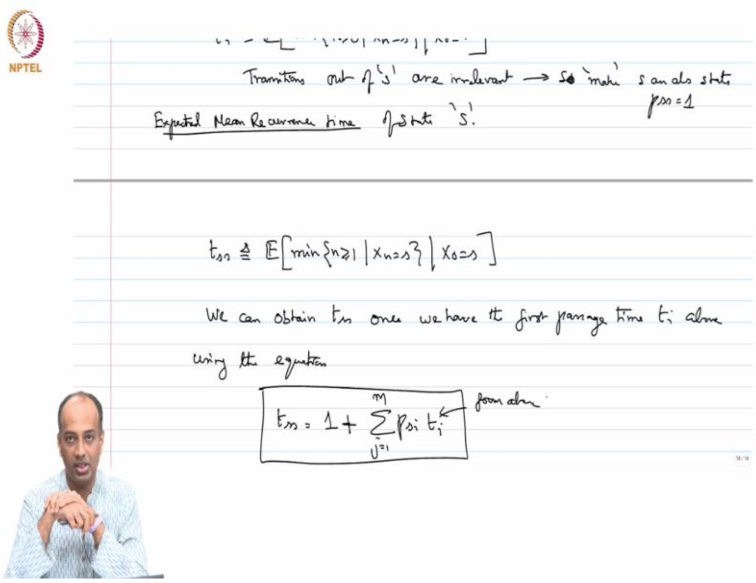
This state s does not have to be absorbing or there is no need like there should be a spider there that we cannot out of. So, let us say that there is some generic state s. You are starting i and you want to calculate the expected time until the first time you reach s. Now, what you can do is you can make the state s an absorbing state even if in the original Markov chain if state s need not be absorbing you are just looking at the expected first passage time.

So, what you do is you make the state s an absorbing state and you calculate the expected time to absorption. Then you can get the expected time to go from i to s. Expected first passage time or mean first passage time is the let us call this t i let s be any state not necessarily absorbing and X 0 equals i. t i is defined as the expected min n greater than or equal to 0 such that X n equals s, So this is the first time you reach s given X 0 equals i.

So, we are not saying necessarily that s is an absorbing state, but we are only interested in the first time the first passage to s starting at i. So, then what you can do is you can make s an absorbing state. So, in this case transitions out of s are irrelevant. So, make s an absorbing state and then you apply the formula that we derive for the mu above. So, essentially we make how do you make s an absorbing state you make p ss = 1 and then you apply the formula above.

So, you are given some Markov chain and you are looking at the expected time until I first reach a state s starting at i. So, what I do I modify the Markov chain to make p ss = 1 and then apply the formula derived above for this fly example the mu formula and you get the expected first passage time. Also I want to mention expected mean recurrence time of state s. What is the expected mean recurrence time of state s?

(Refer Slide Time: 25:41)



It is defined as t ss is defined as expected number of transitions such that X n equals s given X 0 equals s. So, you are starting at some state s what is the first time after that that you return back to s and what is the expectation of such a time. So, this is simply the first passage time not starting at i, but this starting at state s itself. So, how do we calculate t ss? We can obtain t ss once we have the first passage times t i above using the equation.

T ss = 1 + so you are starting at s. See now you cannot afford to make s an absorbing state. So, in the earlier we could say I am starting at i I want to calculate the min passage time to some other state s. I could just forget about transitions out of s and make s a absorbing state and then I could t i using the formula above, but if s you are starting at s then you have to transition out of s into some other state and then come back to s.

So, now you cannot afford to make s an absorbing state. So, what you do is to calculate the expected mean recurrence time of state s you first calculate the t i which are the first passage time starting at I then you say from s I have to go to some i and then come back to s. So, from s I have to go to i probability of that is p si and I have the min first passage time from i to s t i.

So, this is just i j = 1 to M of course i can be s itself. So, this is the equation. So, if know t i you know from above so how did you calculate t i? You make s an absorbing state calculate the first passage time from i to s then a recurrence time from s to s mean recurrence time can be calculated using this formula. So, these are useful in calculating these expected mean recurrence time, expected time to capture in these kind of problems and you can work out several examples along these lines. So, this concludes my discussion of short term behaviour. Thank you.