Stochastic Modeling and the Theory of Queues
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Lecture –50
Stationary Distribution and Long term Behaviour of a DTMC - Part 3

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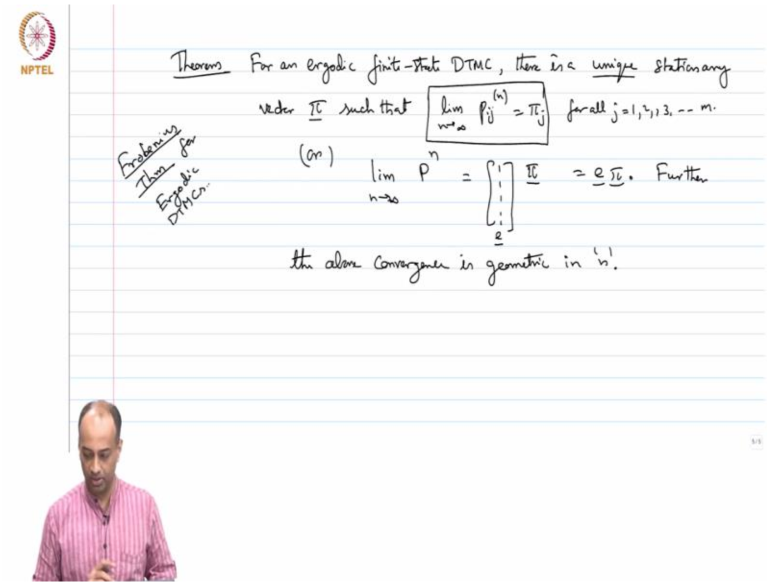

So theorem; let me just go to state whatever we have arrived for an ergodic finite state DTMC there is the unique stationary vector pi such that limit n tending to infinity P ij n = pi j for all j equals 1 to m or this can be rewritten as limit n tending to infinity P to the n is equal to we will write as 1, 1, 1, 1, 1 times the row vector pi. So, if you call this column vector of 1s as e this is just e pi the notation of our goal.

So, the n step transition probability matrix converges to e pi, pi, pi the rows of all pi’s. Further, the above converges is geometric in n. What I mean is that this limit has reached geometrically fast in n because we have that 1 – 2 beta over to the n type of a behaviour so that is the result that we have proven. Note that we have not explicitly proved uniqueness which can also be proved fairly in a straightforward fashion. We look at this proof this theorem which I illustrated.

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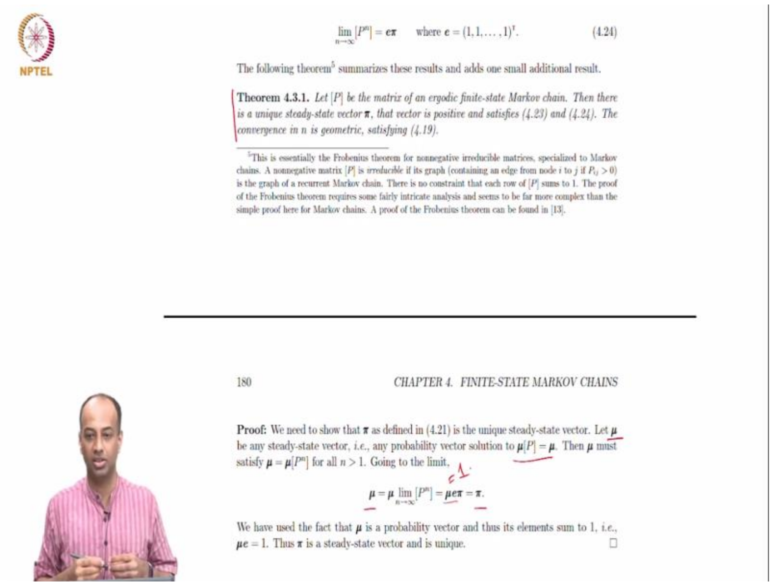

This the theorem that we have stated the theorem that I stated and the proof is here the uniqueness proof. So, if you take any other mu and say that there is another solution mu that you converge to then that mu we know that mu has to satisfy mu P equals mu. This we argued earlier using Chapman–Kolmogorov. So, then you finally end up proving that this step we end up proving that mu has to be equal to pi that is because this mu e = 1 that is the issue.

We just clarify this pi is the limit of the maximum over pi j is the limit n tending to infinity max over i which is equal to limit n tending to infinity min over i. I am just calling this pi j. Is this the same pi j which satisfies pi = pi p the answer is yes that is what is being proved here. So, there is this unique solution so pi = pi P and that is the limit of this P to the n that is what is proved in this theorem.

So, if this theorem is sometimes known as Frobenius' theorem for ergodic Markov chains. So, we have come very close to what we wanted to prove. So far we started with any finite state DTMC. We said that the max over each column is decreasing as n increases and a min over each column is increasing as n increases in the matrix P to the n. Then we consider the special case of a strictly positive matrix P and showed that this decrease and increase are exponentially fast and therefore there has to be a common limit.

Then we considered any ergodic Markov chain not necessarily with P strictly greater than 0 and then we said that some power P to the h must be strictly positive because there are only finitely many states in this ergodic chain then we applied that previous result to the matrix P to the h and we obtained what we want. Basically we prove that P to the n converges to the matrix with all rows pi, pi, pi and this convergence is exponentially fast.

Now, there is only one step remaining in what we actually wanted to prove so what if we have a ergodic unichain what does that mean? You have an ergodic class, but there could be some transient states that is the result that we finally want to prove. So, far we have got as far as having just one recurrent aperiodic class or basically one with that entire Markov chain consists of an ergodic class.

If you allow some more transition transient state then we should still be able to prove it P to the n converges to pi, pi, pi. Now the question is what pi does it converge to? So, if you start in the single recurrent class that you have you are going to be inside the recurrent class and the situation is exactly like the previous situation for what we just proved Frobenius' theorem. However, if we start in a transient state the question is what happens.

Transient states by definition you will stay in the transient state only for a finite amount of time. So, you get out of the transient states and get into one of the recurrent states and you just keep going inside the recurrent state. In that situation what happens is in some finitely many steps once you get out then the behaviour is covered by Frobenius' theorem and for transient states you basically have 0 stationary probability that is what happens. So, this is something that we can show. So, if you have an ergodic unichain.

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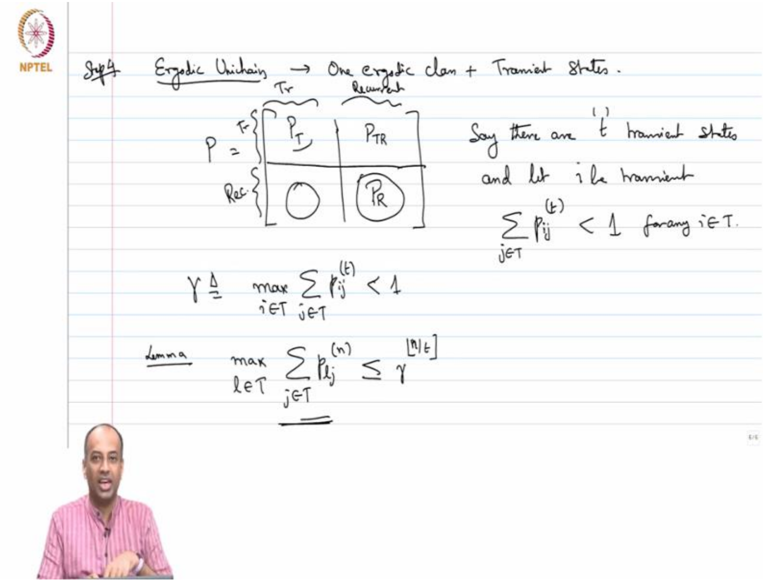

So this is step 4 I guess so we have an ergodic unichain which means that one ergodic class plus some transient states then what happens is the question. So, then I can write out the transition matrix P as follows. See there will be a recurrent component so there will be a component here which is let me call this P R. These are transitions between the recurrent states.

And then there will be let us say these are so I am saying that these states are all transient and then that these states are all recurrent. These are transients and these (()) (08:21) are recurrent. So, let us say this P T these are transitions between transient states and this is transition between transient states to recurrent states. However, this part of the matrix has to be 0. Why because you cannot go from a recurrent state over transient state it is against the definition of a recurrent state.

So, this will be the structure of the transition probability matrix of an ergodic unichain with some potential transient states. Now what happens is so this part is covered by Frobenius theorem. Now we just have to show that if I start out here I will not; if I start out in a transient state I will eventually get to a recurrent state and I will be covered by Frobenius theorem so that is all that I have shown.

So, this can be done as follows. So, what you can do is so let us say there are t transient states and let i be transient. So, the point is so if I start at i which is a transient state there are only t transient states. So, there must be a probability that in t steps there should be a positive probability of getting out of this transient state which means that if I sum over all the transient states i is transient remember the probability of starting at a transient state i and still remaining in one of the transient states must be strictly less than 1.

There is a positive probability there are only t transient states after all so there is a positive probability you are going from i to some recurrent state and (()) (10:49) recurrent state you are never going to comeback. So, this is true for any i which is transient. So, if we look at max over i belongs to T and you take that sum over j belongs to transients P ij t this sum also has to be strictly less than 1.

Let me call this gamma this gamma is something strictly less than 1 then what we can show is a lemma like this. We can show that max over l belongs to transients sum over j belongs to transient P lj n less than or equal to gamma raise to n over t. So, what is this saying is that so this is very easy to prove because I had a probability of less than 1 after t steps that I am still in a transient state is strictly less than 1 which is gamma.

So, I keep giving you a very large n there is a probability that I still remain in a transient state after a very large n so that is what this probability is that has to decay geometrically faster here. This is the lemma that you can easily prove using this property and that if you look at this Gallagher book so it explains the structure of the matrix.

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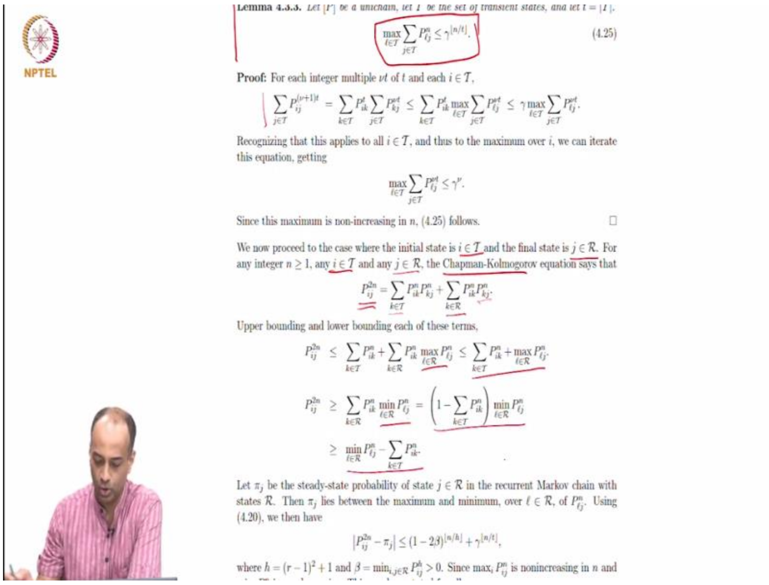

And the lemma I am talking about is this one which is just a pure algebraic manipulation (()) (13:03). Then you can actually go ahead and proof that then you can prove the result you want. Next, you proceed to the case where the initial state is some transient state i and then you go to some recurrent state j in R j which is recurrent. So, what you have proven so far in that lemma is that so in this previous lemma you have shown that the probability of still being transient after a large n number of steps goes down exponentially fast.

So, there is overwhelmingly less probability or overwhelming high probability of being in a recurrent state once we start in a transient state. So, once we get to the recurrent state or a transient state then you can again look at this Chapman–Kolmogorov equation you start with any i which is transient and any j which is recurrent and you look at P ij 2 n which by Chapman–Kolmogorov this is again this is just Chapman–Kolmogorov here you can write out this equation.

You are just splitting the sum over different transient states and different recurrent states. So, then what you do is you have upper bounding and lower bounding. So, here you are just looking at the max so you are just replacing P kj with the max and you get that term and likewise you are just replacing this term with the min and you were getting this kind of a term.

So, you get this lower bound and upper bound on P ij 2 n. Now in the recurrent Markov chain corresponding to P R let pi j be the steady state probability then we know from the previous consideration that this pi j lies between the maximum and min over l of P lj n so this is like previous reasoning. So, you will get P ij 2 n is less than or equal to; so this is like step 3 and this is what we have proven in this lemma here this guy.

So, you will get that the difference between P ij 2 n and pi j decays geometrically fast and therefore you will have convergence. So that is what (()) (15:48). So, we can just take a theorem so what we have proven or rather walked you through this proof (()) (15:56) instead of writing everything out to great detail is that we can prove the following theorem. So, essentially what you are proving is that which we started a transient state you get out of it exponentially fast.

And you get into a recurrent state and after you get into a recurrent state Frobenius theorem applies and then you converge exponentially fast to pi.

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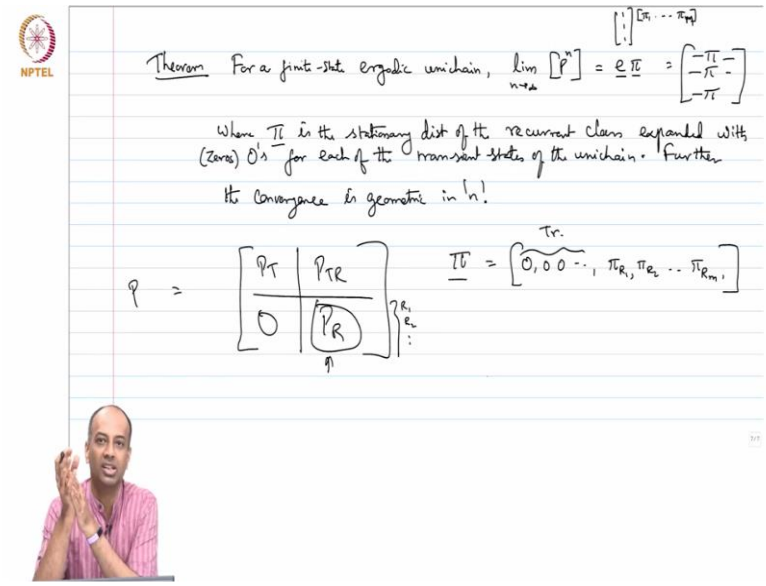

So, the theorem that you can state is the following. This is for ergodic unichain for a finite state ergodic unichain limit n tending to infinity P to the n = e pi. Remember this e is just the column of 1s and pi is just pi 1 dot, dot, dot pi m (()) (17:02). So this is nothing, but your matrix of rows pi where pi is the stationary distribution of the recurrent class expanded with 0s for each of the transient states of the unichain.

Further, the convergence is geometric this convergence is limit n tending to infinity convergence is geometric in n. So, if you go back to the matrix so it look like this if you look at so this was P R, this was P T this was 0 and this is P T R this is how the P looks. So, this P R has a pi vector. So, the pi will correspond to 0, 0, 0 as many as there are transient states and then there will be the pi R.

We can look at this pi R as the vector of pi so you can look at this as now you start with pi R 1, pi R 2 and so on. So, basically I am looking at so these are R 1, R 2 these are the recurrent states so on till pi R I do not know how many (()) (19:40) let us say little m recurrent states that is what happens. So, you have for the matrix P R which is the recurrent component you have a pi and the pi for the entire ergodic unichain will be just 0 augmented wherever there is a transient class.

And the rest of the pi is just copied from the steady state probability or a stationary probability distribution of the recurrent class. And the convergence of course is geometric in n because this showed this equation this equation shows that the convergence is geometric n. So, what we have shown is that if you have a finite state DTMC which is a ergodic unichain. In other words if you have a single recurrent aperiodic class plus some transient states possibly then the long term behaviour of this Markov chain is that P to the n the n step transition probability matrix converges to a steady stationary vector which is solution to pi is equal to pi 1.

So, this is where we have answered k 3 basically in what we wrote down and we stop here for today. Thank you.