Stochastic Modeling and the Theory of Queues

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Module - 1
Lecture - 5
Counting Process

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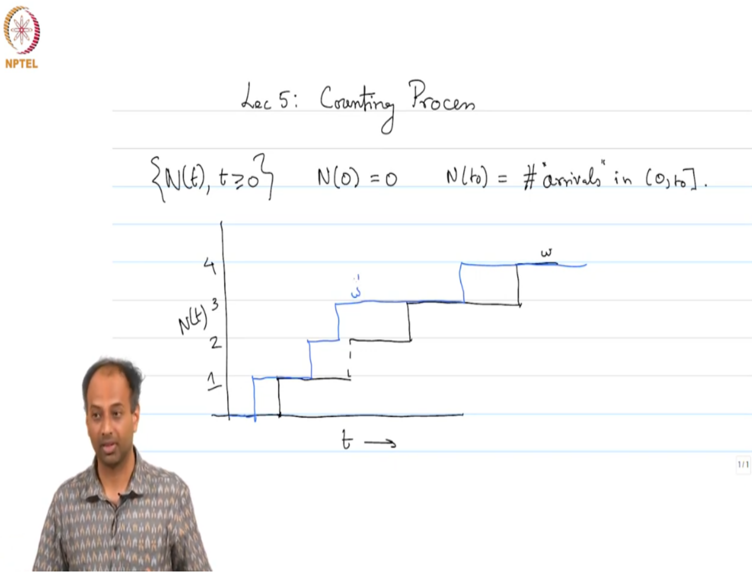

Today we will discuss a counting process. Last lecture, we briefly described what a stochastic process is. We said that a stochastic process is simply a sequence of random variables indexed by time. So, the time variable could be a discrete variable or a continuous variable. So, it could be a discrete time stochastic process or a continuous time stochastic process. This counting process is something we will define now, and we will use this throughout the course.

So, this counting process; we will first look at the, in continuous time. So, we look at non-negative time, N(t). So, you look at N(t). N(0) is taken to be 0. And loosely, $N(t_0)$ is the number of arrivals in $(0, t_0]$. So, you can think of this N(t) as simply counting the number of some arrivals. This arrivals, you could be standing at a bus stop and counting the number of buses.

Or, we could be waiting for a radioactive sample to emit the alpha particles. You could be, you could have a counter and count the number of particles that have been emitted. So, at some time t_0, non-negative time t_0, it counts the total number of arrivals, the values; arrivals could be bus arrivals or radioactive decays or whatever; in the interval $(0, t_0]$; 0 not included; t_0 included. Is that clear?

And this is defined for every t_0 greater than or equal to 0. And this is a random variable. So, for a fixed time t_0, $N(t_0)$ is the random variable. So, this $N(t)$ takes non-negative integer values. So, some particular sample path could be like this. So, there is always 0. This is $N(t)$ against t. So, the first arrival might have occurred here. Then, $N(t)$ jumps by one unit, and then stays constant.

Then, another arrival comes here. It jumps by one more, and so on. 1, 2, 3, 4 and so on. So, this is for a particular realisation ω. So, what I am really plotting here is $N(t, \omega)$. If you had some other realisation or some other little ω, then you could have a different; the step function will look different; it could look like that. Awesome such. This is for some ω'.

So, given different realisations ω, you get different these step functions. And this $N(t)$ is the counting process of interest. Now, there are; so, this $N(t)$ describes a sequence of random variables, for each continuous time t. And for any particular ω, you get a step function $N(t, \omega)$. And for any particular time, this $N(t)$ is some random variable.

That is the picture we have in mind. Now, this counting process can also be described equivalently in terms of the inter-arrival times of these buses or radioactive particles or whatever it is that you are talking about.

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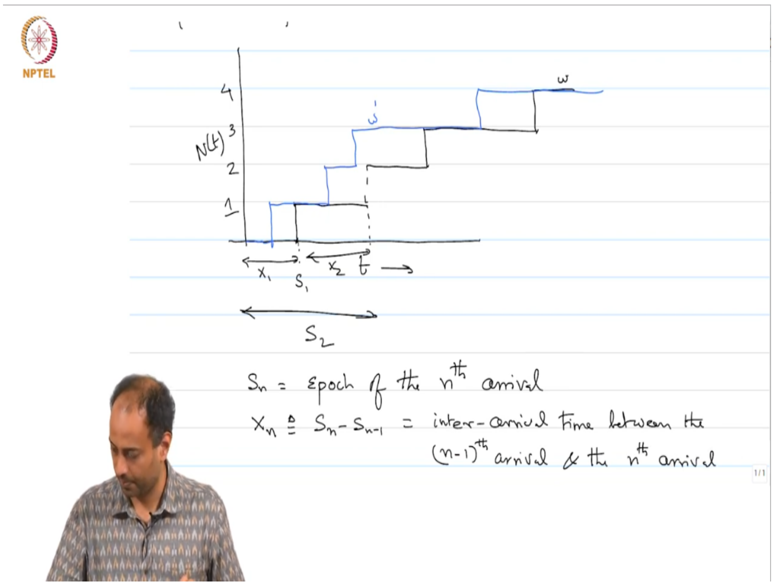

So, if you are looking at the black sample path that I drew here, the first arrival happened, at some particular time, let us say S_1. The second time, second arrival happened at a time; so, let me call this as S_2. And I can call these inter-arrival times as X_1, X_2 and so on. So, this is $X_1(\omega)$, $X_2(\omega)$ and so on.

So, S_i, or S_n denotes the epoch; it is called an epoch. "e p o c h". Epoch of the nth arrival. So, S_2 here is the epoch of the second arrival. So, we can view this S_n. Let me write here. S_n as the epoch of the n^{th} arrival. I am just defining certain things. Likewise, you can call X_n as S_n minus; define X_n as S_n minus S_{n-1}. This is, has the interpretation as, this can be interpreted as the inter-arrival time between the $(n - 1)^{th}$ arrival and the n^{th} arrival. Is it clear?

So, in this picture, S_2 is this total length; S_1 is the epoch of the first arrival; S_2 is the epoch of the second arrival; X_2 is simply S_2 - S_1. So, it is the time between the second arrival and the first arrival. Is it clear? Clearly, there is a one-to-one relationship. So, if I give you the S, the S_i's, by arrival epochs, you can go ahead and calculate the X_n's, using this relationship. Likewise, if I give you the X_n's, the X_i's, you can calculate the S_i's.

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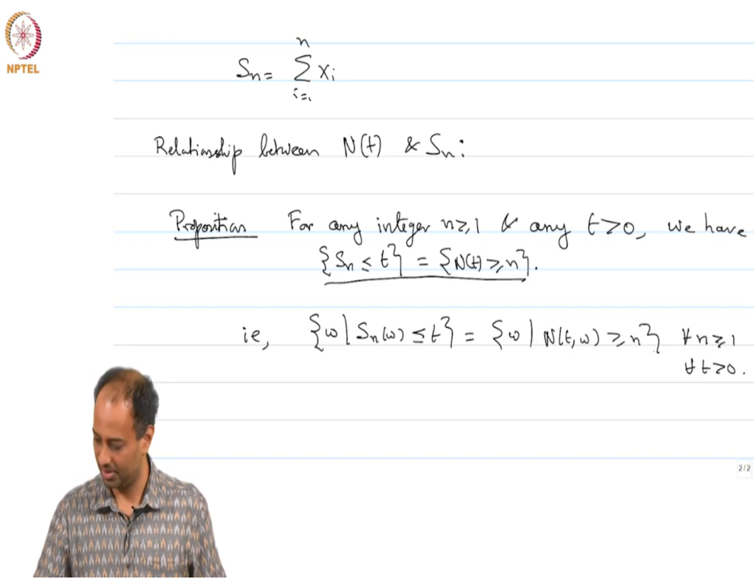

So, you can write, for example, $S_n = \sum_{i=1}^{n} X_i$.

So, you can go back and forth between the sequence X_i's and the sequence S_i's. Clear? So, these are all random variables. So, given ω realises and this entire sequence X_i's realises, the entire sequence S_i's realises. For different ω's, you get these different realisations. And of course, you get different N(t,ω)..

For different omegas, you get different step functions. Any questions on this? This picture? So, we have talked about a counting process. It is simply a non-negative integer valued process defined in continuous time, which just counts the number of arrivals until time t. And associated with this process N(t) are these two different sequences of random variables; S_n's, these are which are the arrival epochs, and these X_n's which are the inter-arrival times.

Now, there exists a very nice relationship between N(t) and S_n. What can be shown is a relation like this. So, let me say a proposition. For any integer n greater than or equal to 1 and any time t > 0, we have the event that $\{S_n \leq t\} = \{N(t) \geq n\}$. So, for every n and every t; every integer $n \geq 1$ and any $t > 0$, we have that $\{S_n \leq t\} = \{N(t) \geq n\}$.

So, now, let me tell you, I mean, maybe I should write this out a little bit. So, what I really say is this.

$$\{\omega |\, S_n(\omega) \le t\} = \{\omega \mid N(t, \omega) \ge n\},\ \forall n \ge 1,\ \forall t > 0$$

This is what I mean by; when I write that, I really mean that. So, if your realisation ω is such that your $S_n(\omega)$, the epoch of the n[th] arrival is before or at t; it is less than or equal to t; no, it is not after t.

Then, for that ω, we are saying that the number of arrivals until time t, must be at least n and vice versa. So, we are basically saying; so, this is a set, right? This is a set of all ω's; this is a subset of the sample space. This is some other subset of the sample space. We are saying that these 2 subsets of the sample space are equal. What does it mean to say that two subsets are equal?

When you say set A = set B, it means that A is contained in B and B is contained in A. So, in order to prove this, you have to prove that the left-hand side is contained in the right-hand side and right-hand side is contained in left-hand side. So, if your little ω satisfies the left-hand side property, it should also satisfy the right-hand side property and vice versa.

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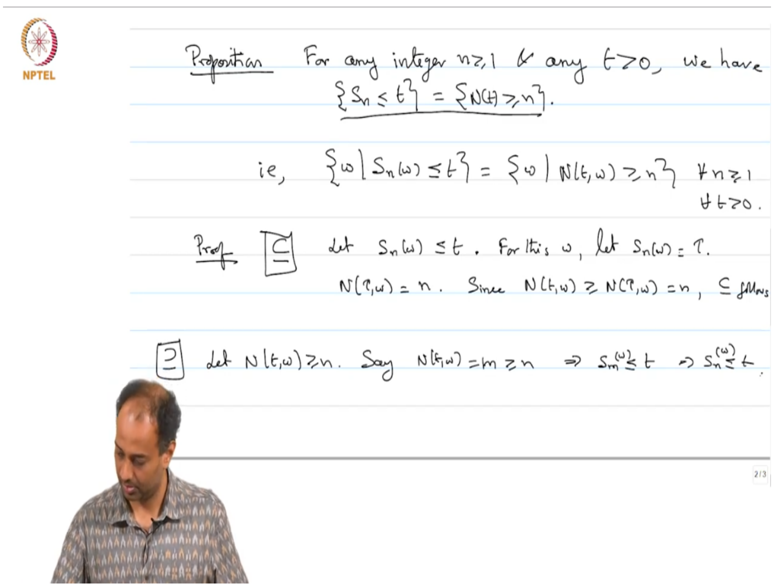

So, you can prove this easily by just going back to the picture. So, first I have to prove, let us say, this containment. So, if I have to prove that containment, meaning that, this guy is contained in that guy; so, I am assuming that; so, let ω be such that $S_n(\omega)$ is less than or equal to t. Clear? So, for this particular ω, $S_n(\omega)$ is some $\tau \le t$. So, for this ω, let $S_n(\omega) = \tau$.

Then, what is $N(\tau)$? For this ω, $N(\tau, \omega) = n$. For this ω, the n^{th} arrival occurred at time τ. So, $N(\tau, \omega) = n$ for this particular ω because you have to count the arrival until that point, including that point. So, this you agree. But it is always the case that if t is bigger than or equal to τ, then N(t) is bigger than or equal to $N(\tau)$. It is obvious through the definition of the counting process.

So, if you have seen a certain number of arrivals till time τ; if you look at a time that is after τ, you cannot have fewer arrivals. That is obvious. So, since $N(t, \omega) \geq N(\tau, \omega) = n$. We are done. So, this guy follows. This containment follows. What have we shown? We have shown that, if $S_n(\omega) \leq t$, then we have shown that $N(t, \omega) \geq n$; which means that this set on the left side is a subset of this set on the right side.

But I have to prove that they are equal, which means I have to prove that the set is contained in this set. So, for the other way round; so, if I want to prove this containment, so, let ω be such that $N(t, \omega) \geq n$. So, what does this mean? For this particular ω, the number of arrivals up to and including time t is at least n, which means that the n^{th} arrival took place. So, we can go back and reason it out.

You can just say that $S_n(\omega) = t$. Because, if the n^{th} arrival took place after t, then N(t) cannot be greater than or equal to n. So, you can say, then. So, maybe you can write this out. So, just to be perfectly clear, this $N(t, \omega) \geq n$; so, let me just do this once properly. So, let us say $N(t, \omega) = m$, which is something greater than or equal to n. So, this implies, $N(t, \omega) = m$ implies that $S_m \leq t$.

Because I have gotten m arrivals at time t, so, the m^{th} arrival has occurred by t, at t or before t. So, this implies that $S_n \leq t$, for that particular ω; maybe, if you want, you write $S_m(\omega)$, $S_n(\omega)$.