

Stochastic Modeling and the Theory of Queues
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Lecture –49

Stationary Distribution and Long Term Behaviour of a DTMC - Part 2

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an ergodic unichain.

By looking closely at the entries $P_{ij}^{(n)}$ as n becomes large.

Step 1 Lemma For any finite state DTMC, for each state j & each integer $n \geq 1$ we have

$$\max_i P_{ij}^{(n)} \leq \max_i P_{ij}^{(n-1)} \quad \text{and} \quad \min_i P_{ij}^{(n)} \geq \min_i P_{ij}^{(n-1)}$$

P^{n-1} P^n

\downarrow \downarrow

[] []

j j

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So, a simple special case is that assume that P greater than 0. Next step so this you can think of a step 1. This is a nice lemma you can think of this as step 1, but it is not quite enough.

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Step 2 Assume $[P] > 0$ $P_{ij} > 0$ for all $i, j = 1, 2, \dots, M$ let $\alpha = \min_{i,j} P_{ij} > 0$

$$\left(\max_i P_{ij}^{(n)} - \min_i P_{ij}^{(n)} \right) \leq \left(\max_i P_{ij}^{(n-1)} - \min_i P_{ij}^{(n-1)} \right) (1 - 2\alpha)$$

$$\Rightarrow \left(\max_i P_{ij}^{(n)} - \min_i P_{ij}^{(n)} \right) \leq (1 - 2\alpha)^n \quad \text{for all states } j \text{ & } \forall n \geq 1.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \max_i P_{ij}^{(n)} = \lim_{n \rightarrow \infty} \min_i P_{ij}^{(n)} \geq \alpha > 0.$$

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The next step is for assume P greater than 0 that means that P_{ij} is greater than 0 for all $i, j = 1, 2, \dots, M$. So, we are looking at now a complete directed graph meaning that you can go from any state i to any state j with strictly positive probability. See this kind of Markov chain is of

course ergodic, but it is a very special case. You are necessarily demanding transition from every pair of states is possible.

You can go from any i to any j for any i and j . This is stronger assumption, but this is a very nice case. So, if this is the situation for this case you can prove the following $\max_i P_{ij}^{n+1} - \min_i P_{ij}^{n+1}$. So, we are going to show that the $\max - \min$ actually goes down geometrically fast. So, let me define let α equals $\min_{i,j} P_{ij}$. So, this α is greater than 0.

Remember that each p_{ij} is strictly positive. So, if we go over all the columns and all the rows look at the smallest element of your transition probability matrix that will still be a strictly positive number we call that number α . So, what we are going to say is that if we look at this difference $\max - \min$ is less than or equal to \max so if n becomes 1 bigger the difference between the \max and the \min reduces geometrically fast $1 - \alpha$ $1 - 2\alpha$.

So, this $1 - 2\alpha$ is sum number less than 1. So, the difference between the maximum entry and the minimum entry in the $n + 1$ step transition probability matrix reduces is a multiplicative factor of $1 - 2\alpha$ compared to the difference between the \max and the \min in the n step transition probability. So, this implies that we have to show of course this implies that if we look at this $\max - \min$ this guy goes down geometrically fast.

So, these both are true for all states j and for all n greater than or equal to 1. So, this is a very strong statement. So, if we have every entry positive the $\max - \min$ the difference between the \max and the \min in any column goes down geometrically fast as n increases. So, as n tends to infinity the difference between the maximum and the minimum goes down to 0 exponentially fast so you must have a convergence.

So, this also implies $\lim_{n \rightarrow \infty} \max_l P_{lj}^n$ is equal to the $\lim_{n \rightarrow \infty} \min_l P_{lj}^n$ and if all of these will be greater than or equal to α because the entries are all greater than or equal to α and this will be greater than 0 this also we can proof α is of course greater than 0. So, in this case so in the previous lemma for any finite state DTMC we proof that the \max over any column keeps decreasing.

And the min over any column keeps increasing, but as n tends to infinity it is possible that they may not actually converge where there may still be a gap. However when the P matrix is strictly positive each entry is positive this gap goes down exponentially geometrically fast. So, the max and the min converge to the same limit and this will be obviously be some function of j and in fact that will be your pi j so this is something we can prove. So, with this special case you can prove that P to the n converges to some pi j.

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NOTE: I AM NOT TRYING TO ARGUE THAT THE LIMIT IS THE SAME AS IN PREVIOUS.

Proof: For each i, j, n , we use the Chapman-Kolmogorov equation, (4.7), followed by the fact that $P_{ij}^n \leq \max_k P_{ik} P_{kj}^{n-1}$ to see that

$$P_{ij}^{n+1} = \sum_k P_{ik} P_{kj}^n \leq \sum_k P_{ik} \max_l P_{lj}^n = \max_l P_{il} P_{lj}^n \quad (4.10)$$

Since this holds for all states i , and thus for the maximizing i , the first half of (4.9) follows. The second half of (4.9) is the same, with minima replacing maxima, i.e.,

$$P_{ij}^{n+1} = \sum_k P_{ik} P_{kj}^n \geq \sum_k P_{ik} \min_l P_{lj}^n = \min_l P_{il} P_{lj}^n \quad \square$$

For some Markov chains, the maximizing value for each column decreases with n and reaches the same limit as that of the minimizing value, i.e., for each column, all elements in that column are the same. This means that each row converges to the same limiting row vector. For other Markov chains, the maximizing value for a column might converge with n to a limit strictly above the limit of the minimizing elements. Then $[P^n]$ does not converge to a matrix where all rows are the same, and might not converge at all.

The following three subsections establish the above kind of convergence (and a number of subsidiary results) for three cases of increasing complexity. The first assumes that $P_{ij} > 0$ for all i, j . This is denoted as $[P] > 0$ and provides a needed step for the other cases. Ergodic Markov chains constitute the second case, and ergodic unichains the third.

4.3.2 Steady state assuming $[P] > 0$

Lemma 4.3.2. Let the transition matrix of a finite-state Markov chain satisfy $[P] > 0$ (i.e., $P_{ij} > 0$ for all i, j), and let $\alpha = \min_{i,j} P_{ij}$. Then for all states j and all $n \geq 1$:

$$\left(\max_l P_{lj}^{n+1} - \min_l P_{lj}^{n+1} \right) \leq \left(\max_l P_{lj}^n - \min_l P_{lj}^n \right) (1 - 2\alpha) \quad (4.11)$$

$$\left(\max_l P_{lj}^n - \min_l P_{lj}^n \right) \leq (1 - 2\alpha)^n \quad (4.12)$$

$$\lim_{n \rightarrow \infty} \max_l P_{lj}^n = \lim_{n \rightarrow \infty} \min_l P_{lj}^n \geq \alpha > 0 \quad (4.13)$$



So, just to walk you through how this is proved if you go to I think I am going back to this Gallager's book. So, this is the statement of the lemma. You are looking at transition probability matrix if each entry is strictly positively and you are looking at the max – min of the n + 1 step transition probability going down by geometric factor 1 – 2 alpha this is what I stated. The proof of this is also somewhat just algebraic and elementary.

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Discussion: Since $P_{ij} > 0$ for all i, j , we must have $\alpha > 0$. Thus the theorem says that for each j , the elements $P_{ij}^{(n)}$ in column j of $P^{(n)}$ approach equality over both i and n as $n \rightarrow \infty$, i.e., the state at time n becomes independent of the state at time 0 as $n \rightarrow \infty$. The approach is geometric in n .

Proof: We first tighten the inequality in (4.10) slightly. For a given j and n , let l_{\min} be a value of l that minimizes $P_{lj}^{(n)}$. Then

$$\begin{aligned} P_{ij}^{(n+1)} &= \sum_k P_{ik} P_{kj}^{(n)} \quad \text{Chapman-Kolmogorov} \\ &\leq \sum_{k \neq l_{\min}} P_{ik} \max_l P_{lj}^{(n)} + P_{i l_{\min}} \min_l P_{lj}^{(n)} \quad P_{i l_{\min}} > \alpha \\ &= \max_l P_{lj}^{(n)} - P_{i l_{\min}} \left(\max_l P_{lj}^{(n)} - \min_l P_{lj}^{(n)} \right) \quad (4.14) \end{aligned}$$

$$\leq \max_l P_{lj}^{(n)} - \alpha \left(\max_l P_{lj}^{(n)} - \min_l P_{lj}^{(n)} \right), \quad (4.15)$$

where in (4.14), we added and subtracted $P_{i l_{\min}} \max_l P_{lj}^{(n)}$ to the right hand side, and in (4.15), we used $\alpha \leq P_{i l_{\min}}$ in conjunction with the fact that the term in parentheses must be nonnegative.

Repeating the same argument with the roles of max and min reversed,

$$P_{ij}^{(n+1)} \geq \min_l P_{lj}^{(n)} + \alpha \left(\max_l P_{lj}^{(n)} - \min_l P_{lj}^{(n)} \right). \quad (4.16)$$

Applying the upper bound, (4.15), to $\max_l P_{lj}^{(n+1)}$ and the lower bound, (4.16), to $\min_l P_{lj}^{(n+1)}$, we can subtract the lower bound from the upper bound to get (4.11).

Next, note that



So, let me just walk you through that so you write out the Chapman–Kolmogorov equation. This is the Chapman–Kolmogorov and so you are replacing; so this sum you are just leaving out the l_{\min} term where l_{\min} is the value of l for which $P_{lj}^{(n)}$ is minimized. So, we are just taking out one term out here of this sum and then you are basically in this step you are adding and subtracting this particular term.

This is just again some algebraic jugglery and so this $P_{i l_{\min}}$. So $P_{i l_{\min}} > \alpha$ is the index where the minimum occurs will be greater than α which means that this inequality follows and similarly if you manipulate for the min so this is the max. Similarly if you manipulate for the min you will get this inequality. These are just algebraic steps. So, you can just subtract the 2 from these inequalities you can just get the first inequality you wanted to proof that there is a geometric reduction in the difference.

And then from monotonicity and the fact that the gap is decreasing exponentially fast you can proof that the limit there is a limit and the limit is strictly positive. So, this is for a Markov chain in which all the entries are strictly positive, but that is too strong an assumption. In fact, we want to prove that P to the n converges to π , π , π for any ergodic unichain, but this is we have assumed that all the states being if the transition between any two pair of states is positive. It is too strong an assumption.

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Step 3 Ergodic chain ← One aperiodic recurrent class + no transient classes.

Recall $[P^n] > 0$ for $n \geq (M-1)^2 + 1 \triangleq h$.

Take the matrix $[P^h]$ and apply the result from step 2.

Let $\beta = \min_{i,j} p_{ij}^{(h)} > 0$

Lemma $\left(\max_i p_{ij}^{(h(m))} - \min_i p_{ij}^{(h(m))} \right) \leq \left(\max_{i,j} p_{ij}^{(h(m))} - \min_{i,j} p_{ij}^{(h(m))} \right) (1-2\beta)^m$

$\Rightarrow \left(\max_{i,j} p_{ij}^{(h(m))} - \min_{i,j} p_{ij}^{(h(m))} \right) \leq (1-2\beta)^m$



So, in step 3 we will relax this assumption. So, you will look at ergodic chain. So, ergodic chain means one aperiodic recurrent class plus no transient classes. We will put the transient classes later. So, we are going from P strictly greater than 0 to P corresponds to a single ergodic class and no transient classes. So, for this recall that for such a ergodic chain P to n is strictly greater than 0 for n greater than or equal to M - 1 whole square + 1.

So, this is a result that we stated I think module 2 ago. If you in an ergodic chain you can go from any state to any other state with positive probability in a finite number of steps. So, if you take n greater than M square roughly then the matrix P power n the n step transition probability matrix will have all entries strictly positive. So, what we are saying is that the matrix P may not correspond to the complete directed graph.

Meaning that some transitions may not be possible, but for n large enough the n step transition probability matrix will have strictly positive entries especially when you take n greater than or equal to this M - 1 square. So, let me call this number equal to sum h. Now, what happens is that if I take so you take the matrix approaches the following. Take the matrix P to the h and apply the result from step 2.

What does that mean? This matrix has strictly positive entries P to the h. So, the matrix P to the h you can apply the result that we proved here that for the matrix P to the h if you take P to the h to the n + 1 and P to the h to the n then this result on your screen will apply. So, P to the h will satisfy this result. So, from this you can work back and proof something for P itself. So, what you can prove is the following.

So, this is the matrix P to the h is strictly positive. Let β equal to $\min_{i,j} P_{ij}^h$ so P_{ij}^h is strictly positive for every i, j and you are looking at this smallest entry in the P to h and so β is strictly positive. So, for this β you can stay a similar result so if you look at $\max_{i,j} P_{ij}^h$ so it will still be a lemma. So, this is a lemma for ergodic chains $\max_{i,j} P_{ij}^h$ to the h times $m + 1 - \min_{i,j} P_{ij}^h$ to the $m + 1$.

This is less than or equal to the same thing with $m + 1$ replaced with $h m - \min_{i,j} P_{ij}^h$. So, we are just pretending that P_{ij}^h has they are strictly positive. So, we are looking at the matrix P to the h and applying the same result so this times $1 - 2\beta$. This is exactly what we know from step 2 in applying the step 2 lemma. So, this implies that this $\max_{i,j} P_{ij}^h$ to the $m - \min_{i,j} P_{ij}^h$ to the m this goes down exponentially fast. So, this will be like $1 - 2\beta$ over to the m .

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Take the matrix $[P^h]$ and apply the result from Step 2.

Let $\beta = \min_{i,j} P_{ij}^h > 0$

Lemma $\left(\max_i P_{ij}^{(h(m+1))} - \min_i P_{ij}^{(h(m+1))} \right) \leq \left(\max_l P_{lj}^{(hm)} - \min_l P_{lj}^{(hm)} \right) (1 - 2\beta)$

$\Rightarrow \left(\max_l P_{lj}^{(hm)} - \min_l P_{lj}^{(hm)} \right) \leq (1 - 2\beta)^m$

$\lim_{m \rightarrow \infty} \max_l P_{lj}^{(hm)} = \lim_{m \rightarrow \infty} P_{lj}^{(hm)} \geq \beta > 0$.


We know $\max_i P_{ij}^{(n)}$ & $\min_i P_{ij}^{(n)}$ are both monotonic


And therefore you can do that limit m tending to infinity $\max_{i,j} P_{ij}^h$ to the m is equal to limit m tending to infinity $\min_{i,j} P_{ij}^h$ to the m is sum number which is strictly positive, but we know that $\max_{i,j} P_{ij}^h$ is strictly positive, so we are now going to the limit along $h m$. We are not able to say limit n tending to infinity. So, we are looking at a subsequence like $h m$ and along the subsequence we are able to show that there is a limit.

But we know that $\max_{i,j} P_{ij}^n$ and $\min_{i,j} P_{ij}^n$ are both monotonic this is the first step we proved and therefore these guys must have a limit as n tends to infinity and that limit

must be equal to the limit along the particular subsequence h_m so then your result follows dot, dot, dot, but this is also there in Gallager.

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$$\min_i P_{ij} \geq \alpha > 0 \quad \max_i P_{ij} \leq 1 - \alpha.$$

Thus $\max_i P_{ij} - \min_i P_{ij} \leq 1 - 2\alpha$. Using this as the base for iterating (4.11) over n , we get (4.12). This, in conjunction with the monotonicity property of (4.9), shows not only that the limits in (4.13) exist and are positive and equal, but that the limits are approached geometrically in n . Finally, since $\min_i P_{ij}^n$ is monotonic non-decreasing in n and $\min_i P_{ij} \geq \alpha$, we see that the limits in (4.13) are lower bounded by α . \square

4.3.3 Ergodic Markov chains

Lemma 4.3.2 extends quite easily to arbitrary ergodic finite-state Markov chains. The key to this comes from Theorem 4.2.4, which shows that if $[P^n]$ is the matrix for an M state ergodic Markov chain, then the matrix $[P^h]$ is positive for any $h \geq (M-1)^2 + 1$. Thus,

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choosing $h = (M-1)^2 + 1$, we can apply Lemma 4.3.2 to $[P^h] > 0$. For each integer $m \geq 1$,


$$\max_i P_{ij}^{h(m+1)} - \min_i P_{ij}^{h(m+1)} \leq \left(\max_i P_{ij}^{hm} - \min_i P_{ij}^{hm} \right) (1 - 2\beta)$$

$$\left(\max_i P_{ij}^{hm} - \min_i P_{ij}^{hm} \right) \leq (1 - 2\beta)^m \quad (4.17)$$

If you look at this bit on ergodic Markov chains. So, you are looking at for any h greater than or equal to that much we are guaranteed that P to the h is positive I have just taken h equals $M - 1$ square + 1.

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choosing $h = (M-1)^2 + 1$, we can apply Lemma 4.3.2 to $[P^h] > 0$. For each integer $m \geq 1$,

$$\max_i P_{ij}^{h(m+1)} - \min_i P_{ij}^{h(m+1)} \leq \left(\max_i P_{ij}^{hm} - \min_i P_{ij}^{hm} \right) (1 - 2\beta)$$

$$\left(\max_i P_{ij}^{hm} - \min_i P_{ij}^{hm} \right) \leq (1 - 2\beta)^m \quad (4.17)$$

$$\lim_{m \rightarrow \infty} \max_i P_{ij}^{hm} = \lim_{m \rightarrow \infty} \min_i P_{ij}^{hm} \geq \beta > 0, \quad (4.18)$$

where $\beta = \min_i P_{ij}^h$. Lemma 4.3.1 states that $\max_i P_{ij}^n$ is non-increasing and $\min_i P_{ij}^n$ non-decreasing in n . Thus (4.17) and (4.18) can be replaced with

$$\left(\max_i P_{ij}^n - \min_i P_{ij}^n \right) \leq (1 - 2\beta)^{\lfloor n/h \rfloor} \quad (4.19)$$

$$\lim_{n \rightarrow \infty} \max_i P_{ij}^n = \lim_{n \rightarrow \infty} \min_i P_{ij}^n > 0. \quad (4.20)$$

Now define $\tau > 0$ by

$$\tau_j = \lim_{n \rightarrow \infty} \max_i P_{ij}^n = \lim_{n \rightarrow \infty} \min_i P_{ij}^n > 0. \quad (4.21)$$

Since τ_j lies between the minimum and maximum over i of P_{ij}^n for each n ,

$$|P_{ij}^n - \tau_j| \leq (1 - 2\beta)^{\lfloor n/h \rfloor}. \quad (4.22)$$

In the limit, then,

$$\lim_{n \rightarrow \infty} P_{ij}^n = \tau_j \quad \text{for each } i, j. \quad (4.23)$$

This says that the matrix $[P^n]$ has a limit as $n \rightarrow \infty$ and the i, j term of that matrix is

So, for the matrix P to the h you can apply this is all from step 2. I am blindly applying for the strictly positive matrix P to the h I am applying that result so this limit along this subsequence h_m I must have the limit, but then I have more atomicity is non-increasing and non-decreasing in n property of P_{ij} sorry \max and \min of P_{ij} n . Therefore, you can just prove that if you go along any n you must have a limit.

And further the max and the difference between the max and min goes down exponentially fast like so. So, now that this limit exists you can just define this common value of limit of the maximum limit of the minimum to be $\sum p_i$ and finally we end up proving that this is π_j lies between the max and the min over the various i .

We must have $|P_{ij}^n - \pi_j|$ goes down exponentially fast. So, this is perfect. So, for an ergodic Markov chain this equation 4.22 which I have just arrived at using this strictly positive matrix P to the h . We have proven this crucial equation 4.22 that $|P_{ij}^n - \pi_j|$ the n step transition probability – π_j which is the limit defined above goes down exponentially fast.

So, not only do you have convergence of P_{ij}^n to this π_j the convergence is exponentially fast geometric $(\rho)^n$ (19:48). So, you have that limit and this limit is approach in a geometrically rapid manner. So, we have proven that for ergodic Markov chain basically a Markov chain in which there is all the states there are recurrent and aperiodic then we have P_{ij}^n converges to this π_j .