Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology – Madras

Lecture –49
Stationary Distribution and Long Term Behaviour of a DTMC - Part 2

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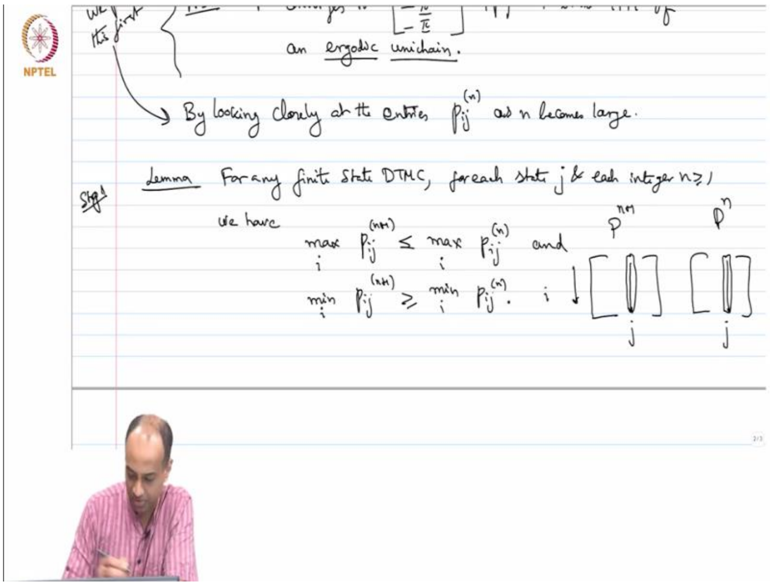

So, a simple special case is that assume that P greater than 0. Next step so this you can think of a step 1. This is a nice lemma you can think of this as step 1, but it is not quite enough.

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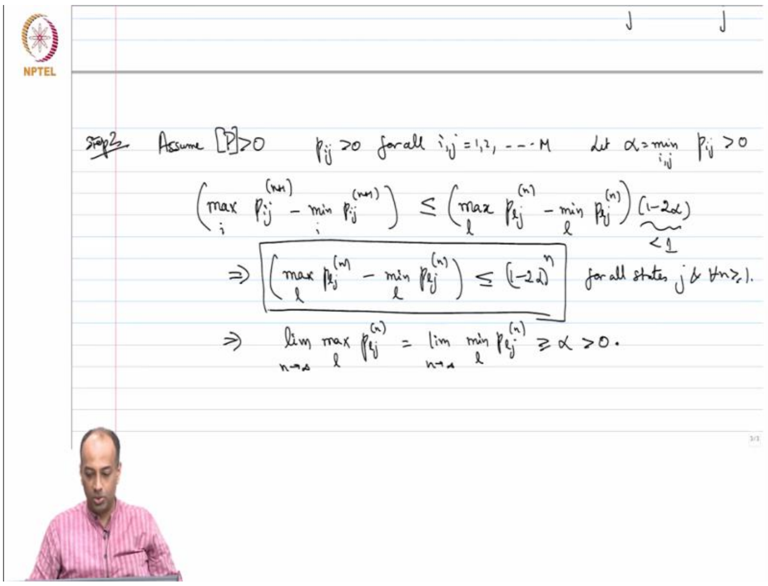

The next step is for assume P greater than 0 that means that P ij is greater than 0 for all i j = 1, 2 M. So, we are looking at now a complete directed graph meaning that you can go from any state i to any state j with strictly positive probability. See this kind of Markov chain is of course ergodic, but it is a very special case. You are necessarily demanding transition from every pair of states is possible.

You can go from any i to any j for any i and j. This is stronger assumption, but this is a very nice case. So, if this is the situation for this case you can prove the following max over i P ij n + 1 – min over i P ij n + 1. So, we are going to show that the max – min actually goes down geometrically fast. So, let me define let alpha equals min over i, j P ij. So, this alpha is greater than 0.

Remember that each p ij is strictly positive. So, if we go over all the columns and all the rows look at the smallest element of your transition probability matrix that will still be a strictly positive number we call that number alpha. So, what we are going to say is that if we look at this difference max – min is less than or equal to max so if n becomes 1 bigger the difference between the max and the min reduces geometrically fast 1 – alpha 1 – 2 alpha.

So, this 1– 2 alpha is sum number less than 1. So, the difference between the maximum entry and the minimum entry in the n + 1 step transition probability matrix reduces is a multiplicative factor of 1 – 2 alpha compared to the difference between the max and the min in the n step transition probability. So, this implies that we have to show of course this implies that if we look at this max – min this guy goes down geometrically fast.

So, these both are true for all states j and for all n greater than or equal to 1. So, this is a very strong statement. So, if we have every entry positive the max – min the difference between the max and the min in any column goes down geometrically fast as n increases. So, as n tends to infinity the difference between the maximum and the minimum goes down to 0 exponentially fast so you must have a convergence.

So, this also implies limit n tending to infinity max over l P lj n is equal to the limit n tending to infinity min over l P lj n and if all of these will be greater than or equal to alpha because the entries are all greater than or equal to alpha and this will be greater than 0 this also we can proof alpha is of course greater than 0. So, in this case so in the previous lemma for any finite state DTMC we proof that the max over any column keeps decreasing.

And the min over any column keeps increasing, but as n tends to infinity it is possible that they may not actually converge where there may still be a gap. However when the P matrix is strictly positive each entry is positive this gap goes down exponentially geometrically fast. So, the max and the min converge to the same limit and this will be obviously be some function of j and in fact that will be your pi j so this is something we can prove. So, with this special case you can prove that P to the n converges to some pi j.

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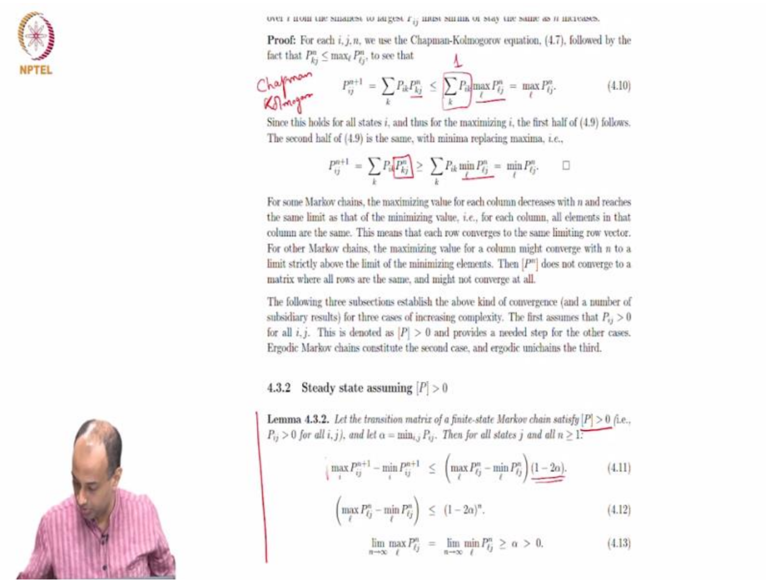

So, just to walk you through how this is proved if you go to I think I am going back to this Gallagher's book. So, this is the statement of the lemma. You are looking at transition probability matrix if each entry is strictly positively and you are looking at the max – min of the n + 1 step transition probability going down by geometric factor 1 – 2 alpha this is what I stated. The proof of this is also somewhat just algebraic and elementary.

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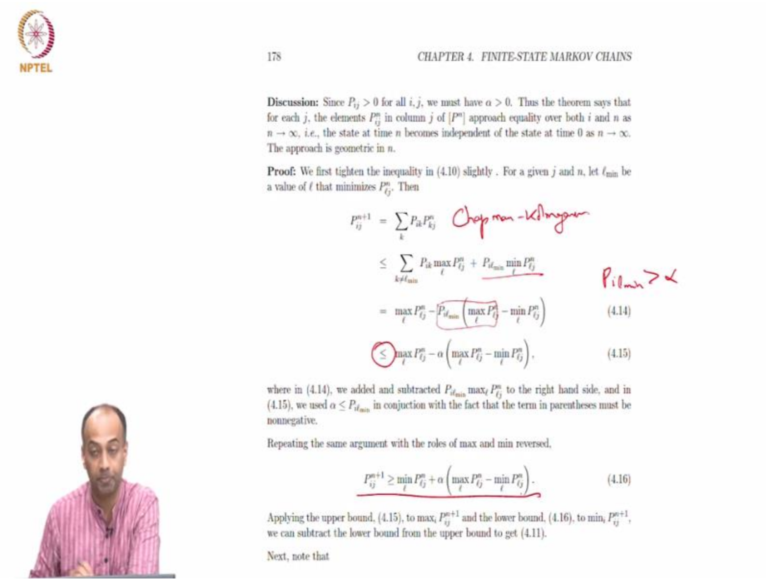

So, let me just walk you through that so you write out the Chapman–Kolmogorov equation. This is the Chapman–Kolmogorov and so you are replacing; so this sum you are just leaving out the l min term where l min the value of l for which P lj n is minimized. So, we are just taking out one term out here of this sum and then you are basically in this step you are adding and subtracting this particular term.

This is just again some algebraic jugglery and so this P il min. So P il min l min is the index where the minimum occurs will be greater than alpha which means that this inequality follows and similarly if you manipulate for the min so this is the max. Similarly if you manipulate for the min you will get this inequality. These are just algebraic steps. So, you can just subtract the 2 from these inequalities you can just get the first inequality you wanted to proof that there is a geometric reduction in the difference.

And then from monotonicity and the fact that the gap is decreasing exponentially fast you can proof that the limit there is a limit and the limit is strictly positive. So, this is for a Markov chain in which all the entries are strictly positive, but that is too strong an assumption. In fact, we want to prove that P to the n converges to pi, pi, pi for any ergodic unichain, but this is we have assumed that all the states being if the transition between any two pair of states is positive. It is too strong an assumption.

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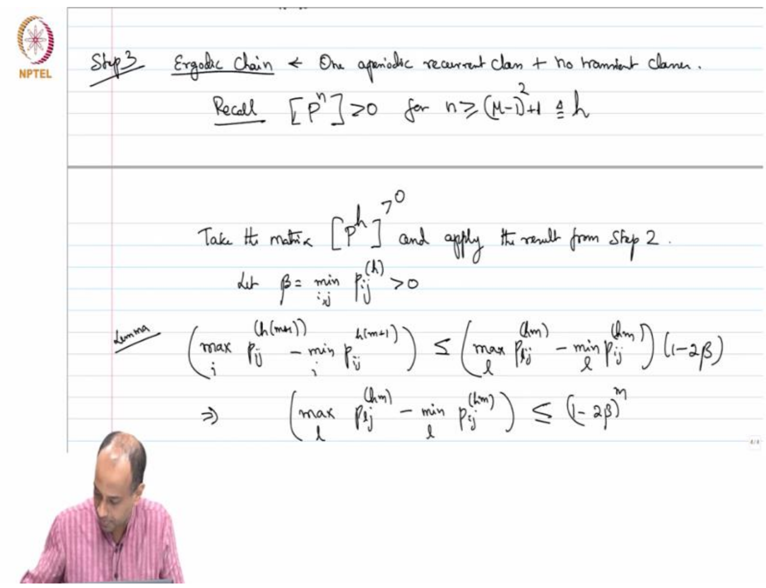

So, in step 3 we will relax this assumption. So, you will look at ergodic chain. So, ergodic chain means one aperiodic recurrent class plus no transient classes. We will put the transient classes later. So, we are going from P strictly greater than 0 to P corresponds to a single ergodic class and no transient classes. So, for this recall that for such a ergodic chain P to n is strictly greater than 0 for n greater than or equal to M – 1 whole square + 1.

So, this is a result that we stated I think module 2 ago. If you in an ergodic chain you can go from any state to any other state with positive probability in a finite number of steps. So, if you take n greater than M square roughly then the matrix P power n the n step transition probability matrix will have all entries strictly positive. So, what we are saying is that the matrix P may not correspond to the complete directed graph.

Meaning that some transitions may not be possible, but for n large enough the n step transition probability matrix will have strictly positive entries especially when you take n greater than or equal to this M – 1 square. So, let me call this number equal to sum h. Now, what happens is that if I take so you take the matrix approaches the following. Take the matrix P to the h and apply the result from step 2.

What does that mean? This matrix has strictly positive entries P to the h. So, the matrix P to the h you can apply the result that we proved here that for the matrix P to the h if you take P to the h to the n + 1 and P to the h to the n then this result on your screen will apply. So, P to the h will satisfy this result. So, from this you can work back and proof something for P itself. So, what you can prove is the following.

So, this is the matrix P to the h is strictly positive. Let beta equal to min over i, j P ij h so P ij h is strictly positive for every i, j and you are looking at this smallest entry in the P to h and so beta is strictly positive. So, for this beta you can stay a similar result so if you look at max over i so it will still be a lemma. So, this is a lemma for ergodic chains max over i P ij to the h times m + 1 – min over i P ij h m + 1.

This is less than or equal to the same thing with m + 1 replaced with h m – min P ij h m. So, we are just pretending that P ij h has they are strictly positive. So, we are looking at the matrix P to the h and applying the same result so this times 1 – 2 beta. This is exactly what we know from step 2 in applying the step 2 lemma. So, this implies that this max over l P lj h m – min over l P ij h m this goes down exponentially fast. So, this will be like 1 – 2 beta over to the m.

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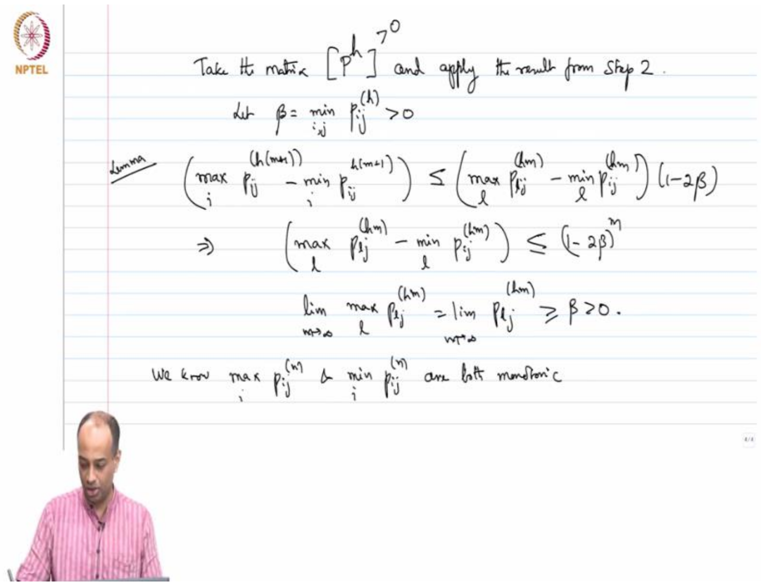

And therefore you can do that limit m tending to infinity max P lj h m is equal to limit m tending to infinity P lj h m is sum number which is strictly positive, but we know that max over i; so we are now going to the limit along h m. We are not able to say limit n tending to infinity. So, we are looking at a subsequence like h m and along the subsequence we are able to show that there is a limit.

But we know that max over i P ij n and min over i P ij n are both monotonic this is the first step we proved and therefore these guys must have a limit as n tends to infinity and that limit must be equal to the limit along the particular subsequence h m so then your result follows dot, dot, dot, but this is also there in Gallagher.

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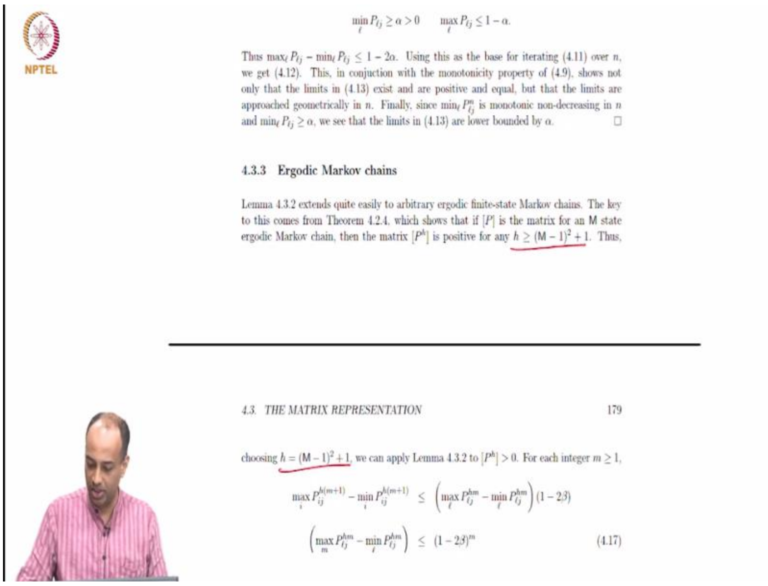

If you look at this bit on ergodic Markov chains. So, you are looking at for any h greater than or equal to that much we are guaranteed that P to the h is positive I have just taken h equals M – 1 square + 1.

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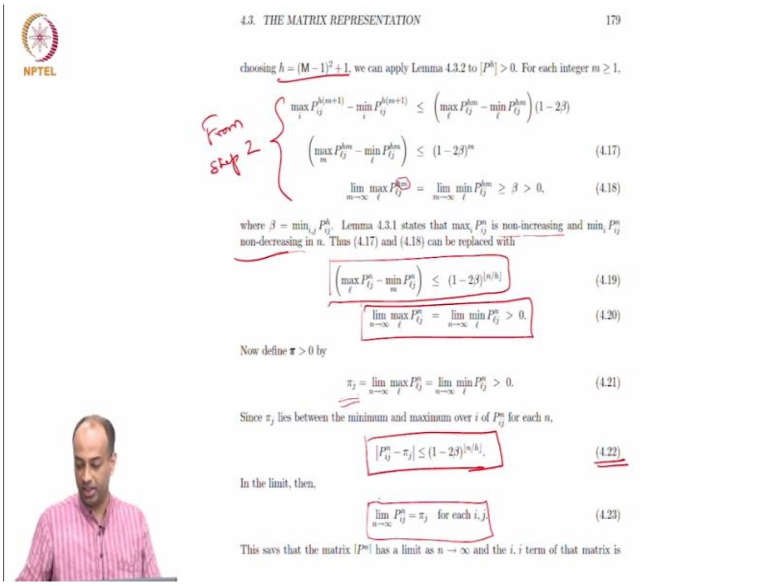

So, for the matrix P to the h you can apply this is all from step 2. I am blinding applying for the strictly positive matrix P to the h I am applying that result so this limit along this subsequence h m I must have the limit, but then I have more atomicity is non-increasing and non-decreasing in n property of P ij n sorry max and min of P ij n. Therefore, you can just prove that if you go along any n you must have a limit.

And further the max and the difference between the max and min goes down exponentially fast like so. So, now that this limit exists you can just define this common value of limit of the maximum limit of the minimum to be sum pi and finally we end up proving that this is this pi j lies between the max and the min over the various i.

We must have P ij n the absolute value of P ij n – pi j goes down exponentially fast. So, this is perfect. So, for an ergodic Markov chain this equation 4.22 which I have just arrived at using this strictly positive matrix P to the h. We have proven this crucial equation 4.22 that P ij n the n step transition probability – pi j which is the limit defined above goes down exponentially fast.

So, not only do you have convergence of P ij n to this pi j the convergence is exponentially fast geometric (()) (19:48). So, you have that limit and this limit is approach in a geometrically rapid manner. So, we have proven that for ergodic Markov chain basically a Markov chain in which there is all the states there are recurrent and aperiodic then we have P to the n converges to this pi.