Stochastic Modeling and the Theory of Queues
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Lecture –48
Stationary Distribution and Long Term Behaviour of a DTMC - Part 1

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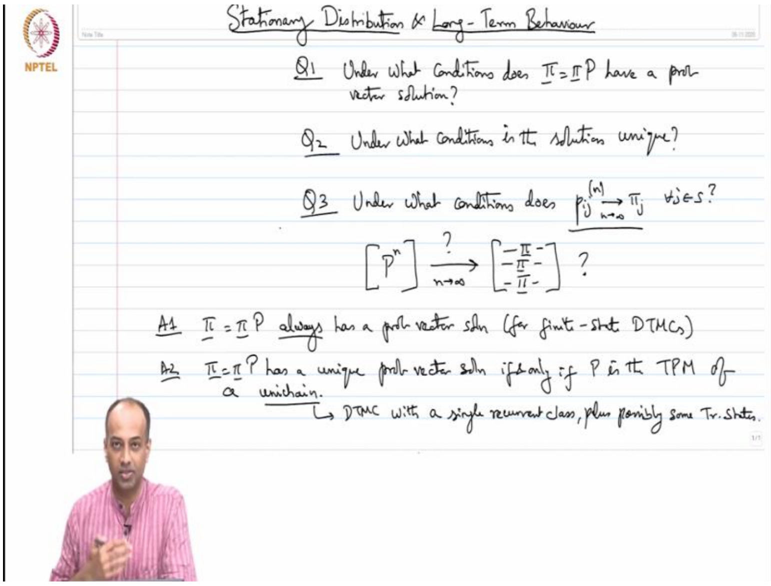

Welcome back. In the previous module we were discussing the concept of the stationary distribution of a finite state DTMC. We said that if there were to exist probability distribution pi that satisfies this equation pi is equal to pi P which is called the global balance equation then if you start in this distribution pi at time 0 then you are guarantee to remain in the distribution pi over the states for all time to come.

In that sense distribution among the states which satisfies the equation pi is equal to pi P is said to be a stationary distribution if such a distribution were to exist. Now we also have to answer we do not know if such a stationary distribution always exist. So, we have to answer some of these important questions that we put down last time under what conditions this pi is equal to pi P have a probability vector solution.

When does such a distribution exist? Question two under what condition is the solution unique, is there a unique solution or they are multiple stationary distributions. These questions we have to answer. We also briefly discussed somewhat informally the conditions for a long term convergence. So, we looked at under what conditions does the n step transition probability P ij n converge to some number pi j irrespective of where you start.

So, we will try to answer these questions in this module and the coming modules. We said that in answering this convergence questions we said that if P ij n were to converge to something which is a function of j that limit has to be in fact the pi j which satisfies pi is equal to pi P this also we indicated we used the Chapman–Kolmogorov equations to show that this P ij n if at all it converges to a limit it has to converge to the solution of pi is equal to pi P.

Of course, we do not know if it converges we just said if it does converge then it has to converge to a solution of pi is equal to pi P. So, there are these questions that we have to answer. So, let me; what I will do is I will put down the answer to these questions and slowly start seeing why these answers are true. So, the answer to the first question is that pi equals pi P always has a probability vector solution for finite state DTMCs.

We can always solve pi is equal to pi P and normalize pi to 1 for a finite state DTMC this may not true for a countably infinite state DTMC. So, we have answered the first question in the affirmative we are saying it always has a solution probability has the solution. Now, is the solution unique? The answer may not be unique so pi equals pi P has a unique probability vector solution if and only if P is the transition probability matrix of a unichain.

So, what is a unichanin? A unichain is nothing, but a Markov it is a finite state DTMC so it is a DTMC with a single recurrent class plus possibly some transient states. So, a unichain is a Markov chain in which there is one recurrent class there may or may not be transient states. Transient states are allowed, but they may not be there also. Remember that in the finite state Markov chain you are guarantee to have at least one recurrent class.

There could be more recurrent classes and there could be other transient states. However, in a unichain, a unichain is a Markov chain in which there is exactly one recurrent class your guaranteed one there could be many, but in a unichain does exactly one recurrent class and there could be transient states there may not be also. So, for a unichain pi equals pi P has a probability vector solution.

So, for unchain if there is only one recurrent class then pi equals pi p has a solution and that is a unique solution. Generally what happens is that if there are multiple recurrent classes say there are k recurrent classes then pi equals pi P will have k linearly independent probability vector solutions to pi equal to pi. So, in that case if there are k recurrent classes for k greater than 1 the solution will not be unique.

So, there will be k linearly independent solutions and all linear combinations of those k independent solutions will also be solutions and so on. So pi equals pi P as has as many linearly independent solutions as the recurrent classes in the Markov chain this can be shown. So, these we will prove later a 1 and a 2 these answers to the first two question we will prove later when we do the spectral properties of the matrix P.

When we look at the Eigen values and Eigen vectors and all that Actually we will look at the convergence questions which is Q 3 which is the long term behaviour question and the answer to this question will start studying the third question little more closely and closely following Gallagher book for this topic.

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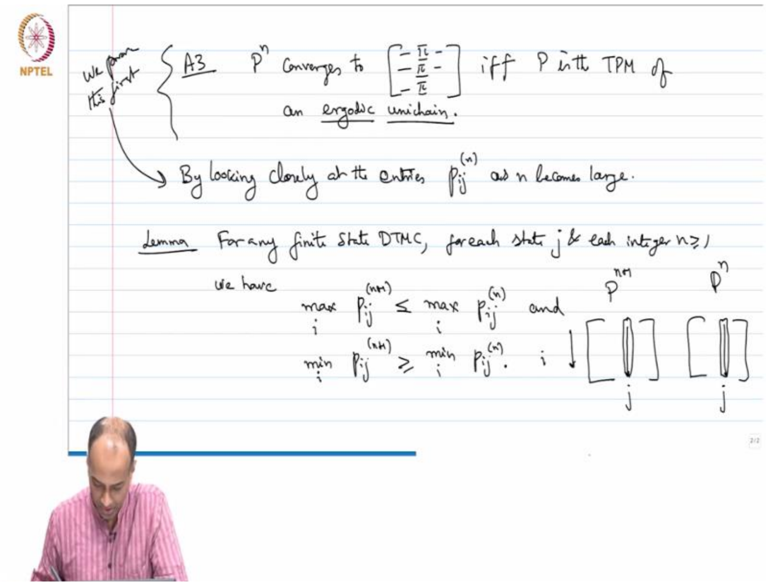

Answer three. So, question three is when does this solution does when does P to the n converge to the matrix of pi, pi, pi, pi. In other words when does P i j n converge to pi j for larger (()) (07:58) tends to infinity the answer is the following. P to n converges to the matrix of all pi, pi, pi if and only if P is the transition probability matrix of an ergodic unichain. So, we are looking at convergence now.

So, P n converges to identical rows of pi, pi, pi where this pi is a unique solution to pi is equal to pi P. So, you have a unchain so this unichain so there is a unique solution to pi is equal to pi P and further we are demanding an ergodic unichain so this is if and only if statement. So, if you have a unichain meaning there is only one recurrent class plus probably some transient classes and the recurrent class is aperiodic.

So, if you have a recurrent aperiodic class which is there is only one recurrent aperiodic class and possibly some transient states then P to the n will converge to pi, pi, pi. I mean you can just take this as a theorem which we will do eventually. So, P to the n converges to pi, pi, pi for an ergodic unichain. What is an ergodic unichain? A Markov chain in which there is a single recurrent aperiodic class plus possibly some transient states.

So, we will gradually build up so we will study this first. We will hit towards proving this first. So, we prove this first because we can actually just look at the mechanics of what happens to the matrix P to the n. We can look at the matrix P multiply it with n times with itself and look at algebraically in a very elementary way looks at what happens to the entries of P ij n and actually conclude that under what condition it converges to 5.5.

And whereas the earlier answers A 1 and A 2 will require a little more knowledge of the Eigen value and Eigen vectors of the matrix P which we will do later. So, just proving A 3 just involves those 3 looking at the matrix P and we do that in a few steps and basically we have a series of Lemma (()) (11:09) theorems to finally prove this. So, essentially how do we this?

So, we prove this first by looking closely at the entries P ij n with ijth entry of P to the n entries as n becomes large. The first lemma that starts (()) (11:56) all this is the following. For any finite state DTMC for each state j and each integer n greater than or equal to 1 we have max over i P ij l + 1 so n + 1 step transient probability they are looking at max over i is less than or equal to max over i P ij n and min over i P ij n + 1 is greater than or equal to min over i P ij.

So, what does this mean? You are looking at; so you are fixing so you are saying this for each j and each n greater than or equal to 1. So, you are looking at two matrices you are looking at let us say this is n + 1 step transient probability matrix and this is the n step transient probability matrix. Now we are looking at for each j so you are looking at the max over i. So, looking at the max over all the rows I was just checking so i is the index of rows. So, this is the index of rows and this is the index of columns.

Let us say you fix a particular column let me say you fix a particular column j the same column j out here and you look at this jth column in P to the n + 1 and P to the n and you are looking at max over i and min over i. So, you are running over this column the jth column and looking at the largest entry in the P to the n + 1 out here and the largest entry in P to the n.

So, what we are saying is that for each of this column j the largest entry decreases as n becomes larger and the smallest entry increases. Well, it may not strictly decrease or increase, but is this (()) (15:35) for the max and min respectively. So, what we are saying is as n becomes larger and larger the largest entry decreases in each column and the smallest entry increases.

Now so this is already a very good because something is monotonic your chances of something converging are actually pretty good. The only issue is that this is not a strict reduction and this is true very generally this is true for any finite state DTMC. So, you have the max entry that is decreasing with n and the smallest entry in each column that is increasing with n.

It may have strictly increases it may just take the same also as it increases. So, these guys have to they have a good chance of going to the same limit, but in some cases they do and some cases they do not, but this is already increasing.

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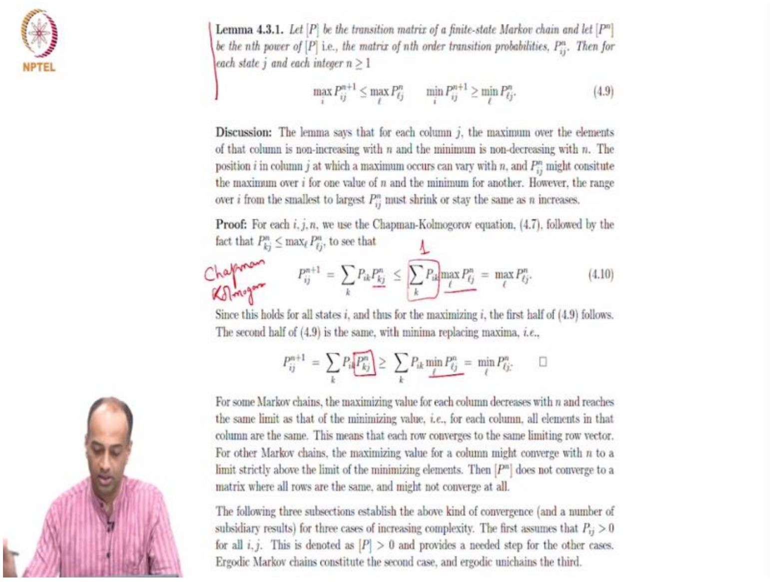

Now, let me just walk you through this is a very fairly easy algebraic sort of a proof comes directly from the Chapman–Kolmogorov equations. So, this time I am just walking you through from how this is proved. So, this is just the statement from the lemma that we just stated from Gallagher. You just write this is Chapman–Kolmogorov equation. This is just Chapman–Kolmogorov equations P ij n + 1 = sum over k P ik P kj n.

And then over here you replace that term with the max over l P l j n then this summation just becomes equal to 1 and you get this inequality. At the reverse inequality is proved by replacing that guy with the min (()) (17:31). So, this is pretty straightforward it is just direct elementary algebraic calculation. So, this is a very easy calculation for showing this result. So, as I said this is good news, but it may not necessarily imply convergence.

For some matrices this limit the max may go to some limit min may go to some other limit and so on, but if you impose a little more if you assume that the Markov chain has ergodicity than you can prove a little more meaning that by ergodic Markov chain we mean that it is both aperiodic and recurrent.