Stochastic Modeling and the Theory of Queues
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Lecture –47
The Long Term Behaviour of a DTMC

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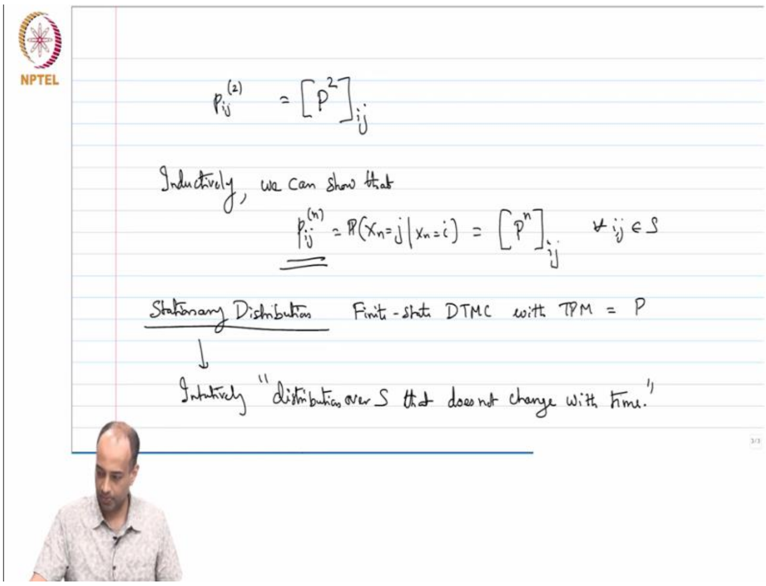

So, this is the matrix notation. Now I want to discuss something known as a stationary distribution. So, I am given a finite state DTMC with transition probability matrix equal to P. Intuitively, stationary distribution is a distribution among state that does not change with time. Stationary distribution is a distribution it is a probability distribution over states over the states space S; let me call over S that does not change with time. So, what does that mean?

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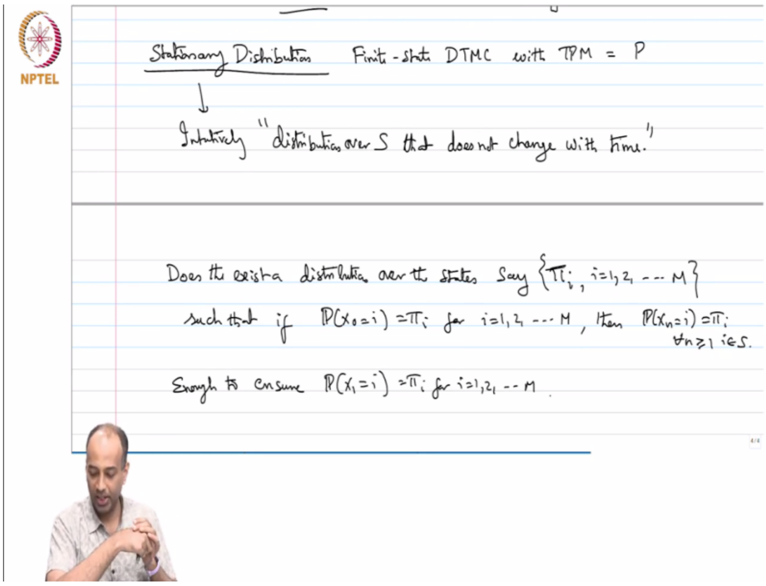

So, we do not know if such a distribution exist. What I mean is that is there a distribution over the states S this is a finite state DTMC. So, is there a probability mass function corresponding to the various states in the finite state DTMC such that if I start in the distribution I always remain in the distribution. Please note that if I start in a particular distribution at time 0.

If it so happens that the distribution at time 1 is the same as the distribution at time 0 then since the Markov chain is homogeneous the distribution at time 1 to time 2 also be the same. So, the question is there a stationary distribution that is an invariant or stationary distribution such that if you start in the distribution in the next step you are in the same distribution. So, that is the question.

Does there exist a distribution over the states. Usually, it is denoted by say pi i; i = 1, 2 dot, dot, dot M such that if probability that X 0 equals i is pi i for i = 1, 2 dot, dot, M then probability that X n equals i is pi i for all n greater than or equal to 1 and i in all the stationary. Now because of the Markov property it is enough to ensure if at all it is possible it is enough to ensure that probability that X 1 is equal to i is equal to pi i.

For i = 1, 2 dot, dot, M because if you start at time 0 in the distribution pi among the states and you remain at pi at time = 1 then because of Markov property the homogenous Markov chain you are going to remain in pi for 2 and 3 and so on. So, it is enough to ensure this.

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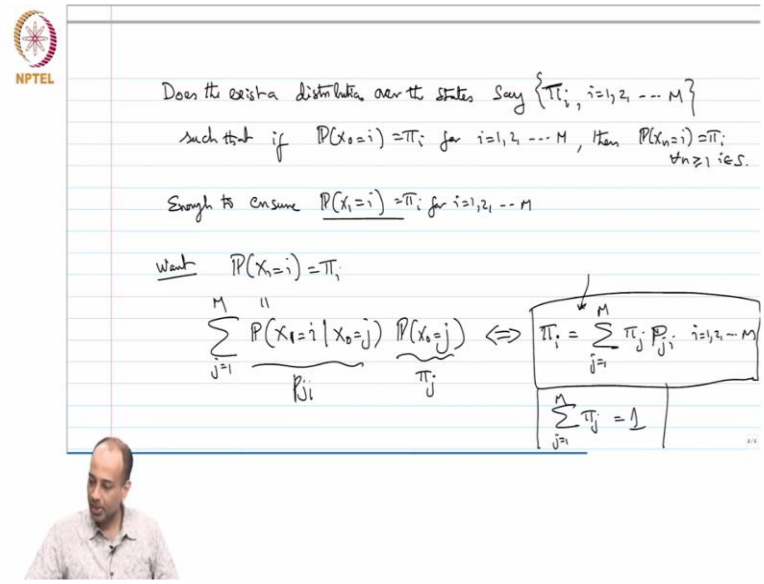

Now, let us look at this. So, we just want to ensure this want probability X 1 equals i equals pi i, but what is probability X 1 equals i. This can be written as probability that X 1 equals i given X 0 equal to j times probability that X 0 equals j by the law of total probability. This is just sum over j equals 1 to M this is the law of total probability. We want to end up the probability that X 1 equals i is simply probability X 1 equals i given X 0 equal to j times probability X 0 equal to j.

But of course you are starting so this equals pi j you want to start at the distribution pi j over states pi i over the states and remain in the pi i. So, what we want is that so this is equivalent to saying what is this equivalent to saying is equal to pi i equals sum over j equals 1 to M. This guy is just P ji this is the jith entry of the transition probability matrix this is pi j P ji. So, this I want for i equals 1 to M.

So, for each state i I want pi i is equal to sum over j equals 1 to M pi j P ji. Of course, we do not know if such a pi even exists and we also want so it is a distribution after all. So, I want to be able to say that it is normalized over the states. The question is does there exist a distribution pi that satisfies this.

If it did I do not know if it does, but suppose I tell you that there is a these M probabilities pi i which of course sum to 1 that satisfy this equation then I am guaranteed that if I start in this pi i at time 0 I will remain at this pi i distribution at time 1 and then therefore at time 2 and therefore for all n. So, if there exist a solution to this a set of equations which I put it in the box here.

Then you are guaranteed that if you start in the distribution you will remain in the distribution for ever. This pi is known as a stationary distribution of the DTMC. I have not said anything about whether such a pi exists all that I am saying is if it did exist it should satisfy these equations. So, you will proceed to study when it exists, when the solution exists, when it is unique and all that.

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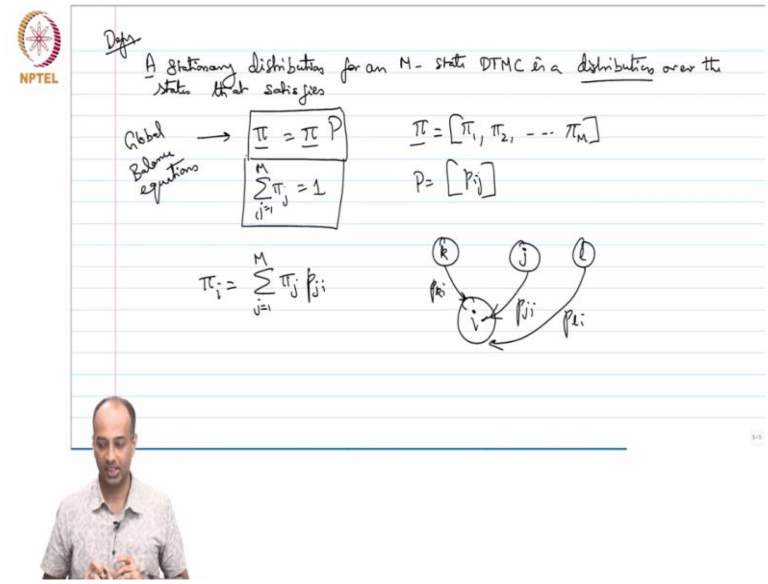

So, let us just say so a stationary distribution for an M state DTMC is a distribution over the states that satisfies. So, I can write this equation right here this equation in the matrix form that is what I am going to do. If it satisfies pi is equal to pi P. So, this pi is a row vector of these probabilities pi 1, pi 2 dot, dot, dot pi M. P is of course the transition probability matrix. So, the equation pi i is equal to sum over j pi j P ji can be written in matrix form as pi is equal to pi P.

And of course pi is probability distribution so it should be normalized I do not have to explicitly say this because it is a distribution over the states, but nevertheless I am putting it down explicitly. So, a normalized vector of numbers which satisfies pi is equal to pi P is known as a stationary distribution of the M state DTMC. Notice that I have said a stationary distribution that I have not said these stationary distribution.

The reason I deliberately put a instead of these because it is possible that there are many stationary distributions as we will see. So, this pi is equal to pi P is a some sort of a cornerstone it is a very important equation when studying finite state Markov chain and also similar thing we will also study for infinite state Markov chains except that P is not a matrix it will be an infinite dimensional entity.

So, these equations are known as the global balanced equations pi is equal to pi P and sum over pi is equal to 1 known as the global balance equations. Intuitively, what does that mean? So, let me just write this down again pi i is equal to sum over pi j P ji j equals 1 to M. So, what we are saying is that the probability of being in some state i is the probability of being in some state j in the previous time and then jumping to state i in the next time.

So, you will do this over all possible states. So, you look at a state i you are looking at the stationary probability of stationary distribution of the state. So, you are looking at the probability that you are in state i where this balance equation is essentially saying that the probability of being in state i is equal to the probability of being in some other state j in the previous time and then jumping to i which is with probability P ji.

And of course you have to go over all the states that you could have been in the previous time. So, it sort of globally balances all the probabilities that is why it is called the global balance equation pi = pi P in matrix form or pi i is equal to sum over j pi j P ji. So, this is the picture to keep in mind. So far we have not said anything about the existence of pi or about its uniqueness.

All we are saying is that if a probability vector pi were to be a stationary distribution it should satisfy pi is equal to pi P and conversely if it satisfies pi is equal to pi P it would be a stationary distribution in the sense that it would be invariant over time. So, there are several questions that arise first does there always exist a solution to pi is equal to pi P existence. The next questions after existence is suppose if it did exist is the solution pi unique?

You can go further even and ask suppose I do not start in the stationary distribution pi suppose I start in whatever distribution I want or I start deterministically in some state. So, I do not care where I start I do not have to start in pi. I start in whatever distribution I want and I let this Markov chain run for a very long time then after very long time if I run this Markov chain for million time steps.

Is it true that the Markov chain after a very long time will the probability of being in state i approach pi i which is the stationary distribution. So, pi = pi P is some sort of a stationary property. Now I am talking about an attraction property. So, if I start off in whatever distribution I want and let the Markov chain run for a long time am I assured that I will attract to this pi after a long time. These are all important questions that we need to understand about finite state DTMCs.

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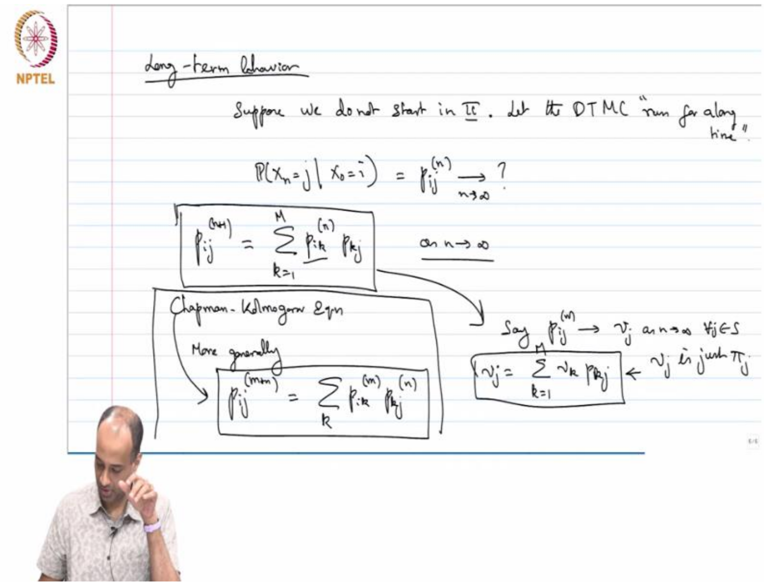

Now, let us discuss long term behaviour. Suppose, we do not start in the steady stationary distribution pi. So, I am going to let the DTMC run for a long time then is it true that the n-step transition probability let us look at P X n equals j given X 0 equal to i which is our old friend P i j n and consider it as n tends to infinity where n is large. What does it converge to? Does it converge at all I do not know and what does it converge to.

Let us look at what it might converge to. So, ideally I would like this P X n equals j given X 0 equal to i to forget where I started if at all it converges to something ideally I would like it to be some function of just j. Suppose, I look at this equation probability so suppose I look at P ij n + 1 I want to go from i to j and n + 1 steps. This is of course is equal to sum over k; k is equal to 1 to M P ik go from i to k in n steps and then go from k to j.

This is I am just rewriting the n + 1 step transition probability matrix the ijth entry in terms of the product of the ijth it is P to the n times P (()) (16:43) P to the n + 1. So, as n tends to infinity what happens to this? By the way this equation is very important equation this is known as the Chapman–Kolmogorov equation of equation. Actually, you can write it more generally I can write Chapman–Kolmogorov equation is written as P i j m + n is equal to sum over k P i k m P k j n.

So, I have to go from i to k in m steps and k to j in n steps. This is actually called the Chapman–Kolmogorov equations. So, anyway so I am back to this considering what happens to this equation as n tends to infinity. So, let me put this in a separate box. I am considering this as n tends to infinity I want P ij n + 1 to converge to sum number which is only a function of j and likewise I want P ik n to converge to sum function of k.

Say I do not know if this happens, but say P i j n converges some nu j as n tends to infinity some number with nu j it is between 0 and 1. Then such a nu j this is for all j in the states so then I would have nu j is equal to sum over nu k P k j k equals 1 to M. Such a nu if it were to converge to a number nu then it would have to satisfy this equation, but this equation is already familiar to us.

It is simply pi is equal to pi P where pi is I am just calling it nu I might as well just call this pi. This is our old friend this is I should why did I call it nu. I could it have just called it i I called it nu because I do not know what it was, but looking at this equation these nu’s are just pi’s j just your old friend pi j. What does that mean? So, if P i j n we have to converge to something some function of j alone where the initial the memory of the initial state is lost then what it converges to P i j n converges to pi j which satisfies pi = pi P.

So, I am talking about two things now. So, one is stationary distribution where if you start in the distribution pi I remain in the distribution pi and such a pi satisfies pi = pi P. So, there you have to ask does there exist such a pi and does such a pi exist uniquely? Next, I am talking about long term behaviour where I do not have to start in pi I start in whatever state I want let us say state i deterministically or whatever distribution I want.

Do I converge to something that forgets where I started? If P ij n were to converge to some function of j some probability that probability has to be pi j which satisfies pi is equal to pi P. So, this is the key issue.

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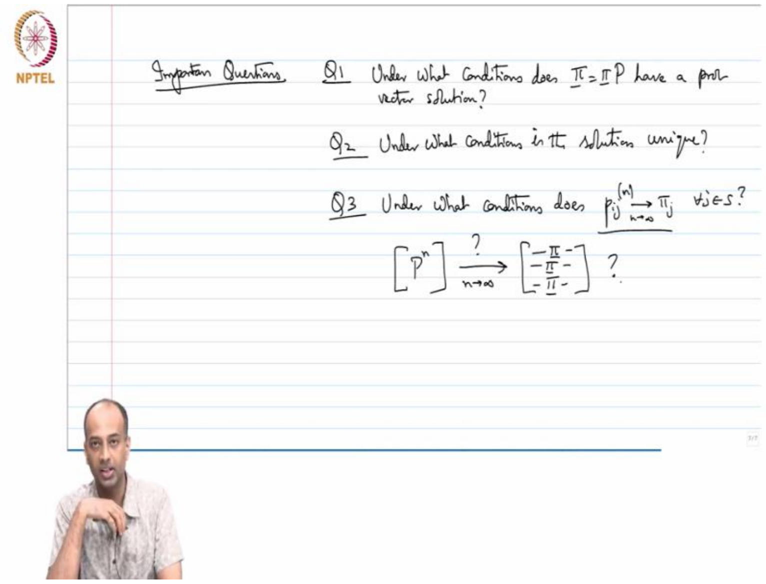

So we can pose some very important questions which we will subsequently answer. Question one under what conditions does pi = pi P have a probability like the solution. Question two under what conditions is the solution unique or non-unique? So, these are existence and uniqueness questions for the stationary distribution. The third question is a long term behaviour question which I eluded to just now.

Under what conditions does P ij n converge to pi j as n tends to infinity for all j, j equals 1 to n or in other words if you look at the nth power of the matrix under what conditions question mark under what conditions does this converge to the vector of all pi, pi, pi what we want is that P ij to go to pi j P ij n to go to pi j. In other words you want the; if you look at the nth power of the matrix P as n becomes large you want all the rows to be pi, pi, pi that is just a matrix version of what I have said here under what conditions does this happen?

So, these are the questions we have to understand about stationary distribution in long term behaviour of finite state DTMCs and we will address these questions in the next few modules. I stop here.