Stochastic Modeling and the Theory of Queues
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Lecture –46
Matrix Representation of a DTMC

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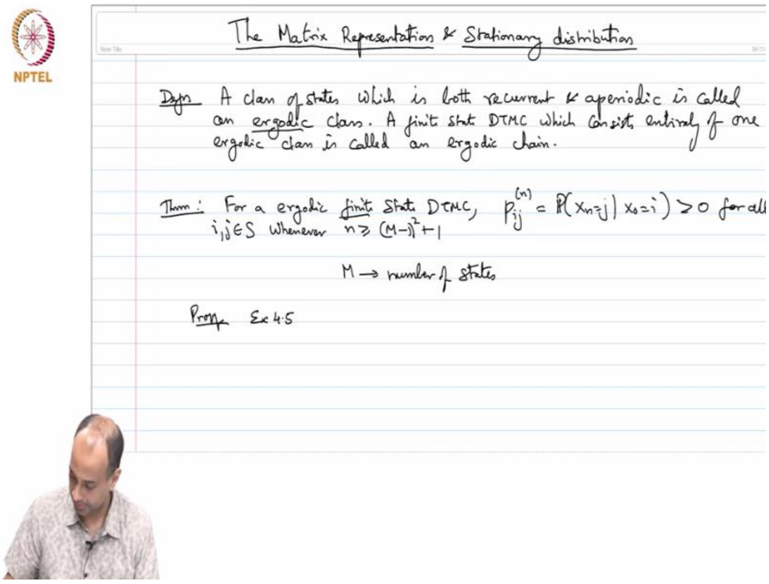

Welcome back. Yesterday we discussed the classification of states into recurrent and transient states or recurrent classes and transient classes and also into periodic and aperiodic states and periodic and aperiodic classes. So, today we will discuss matrix representation and stationary distribution. Before that I want to make an important definition, a class of states which is both recurrent and aperiodic is called an ergodic class.

A finite state DTMC which consists entirely of one ergodic class is called an ergodic chain. This is just definition we know that all states in a class have to either all recurrent or all transient. Similarly, we also know from previous module that in all states in a class have the same period. So, even if one state is aperiodic the entire class is aperiodic. So, if you have a class that is both recurrent and aperiodic in a finite state Markov chain.

Such a class is called an ergodic class and if a finite state DTMC has only one ergodic class then that chain is called an ergodic chain. See these ergodic chains are the nicest kind of Markov chains in some sense in the sense that they have certain steady state long term behaviour which we will study very soon. So, loosely speaking ergodicity means that time averages and ensemble averages are equal.

And we will see that why these aperiodic recurrent Markov chains are called ergodic. It is essentially because after a long time these kind of Markov chains converge to a steady state distribution we will see that in a little bit it is an important result, but for now this is just a definition ergodic is just a name. There is also another definition another theorem I want to state which I probably should have done yesterday in the last module is that for a finite state.

Let us say for an ergodic so if a Markov chain ergodic for ergodic finite state DTMC the P ij n the n-step transition probability which if you recall is probability that X n = j given X 0 equals i. So, this is strictly greater than 0 for all i and j in the state space whenever n is greater than or equal to M – 1 whole square plus 1. Here this big M is the number of states. So, this is a finite state DTMC ergodic finite state DTMC where M is the total number of states big M is some finite number.

What we are saying is that for n large enough when n is roughly like M squared when n is bigger than M – 1 square + 1 then you are guaranteed that the n-step transition probability is positive for every i and j. So, there is a positive probability of going from i to j in n step when n is like M square and the squared of the number of states. So, this basically means that in an ergodic chain you can go from any state to any other state after some amount of time.

There is a positive probability of doing so and this greater than or equal to can be met with equality for some Markov chains. So, the proof of this I am not going to do there is guided proof given in Gallagher exercise 4.5 gives a guided proof for this may not be worth spending class time on this. So, this can be proved and this greater than or equal to M – 1 square + 1 can be met with equalities for some Markov chains.

So, this is a tight bound. So, to summarize ergodic chain is a chain that consists of one recurrent and aperiodic class and for such ergodic chain you have P ij n the n-step transition probability is strictly positive for large enough n when n is like M square.

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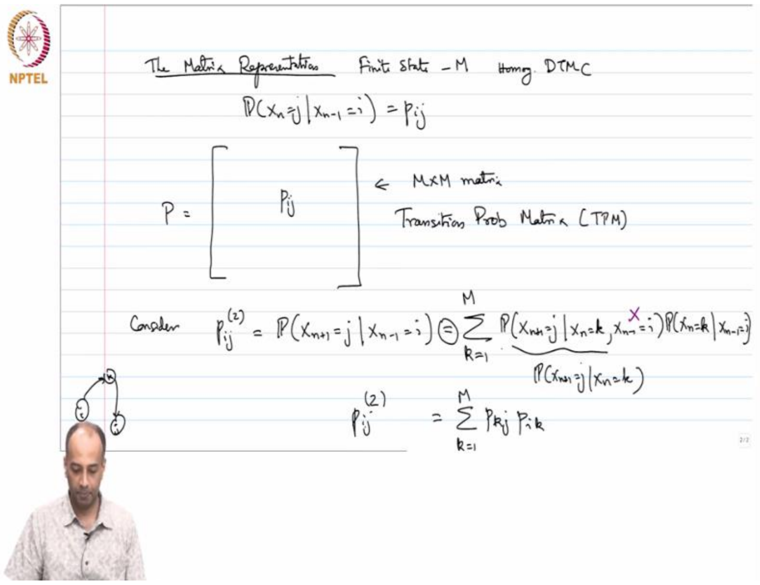

Now let us proceed to discuss the matrix representation. So, we are considering a homogeneous DTMC. Let us say finite state homogenous DTMC; finite state M state homogeneous DTMC. So, what you are given is that P X n equals j given X n – 1 equals i is some P ij. So, this P ij is the transition probability for going from i to j. Now, you define a matrix big P in which this is a M / M matrix in which the ijth entry is simply P ij.

Some of these P i j could be 0 if there is no directed edge between i and j you will enter 0 in this matrix. If not you will enter the value of P ij. So, as i runs through 1 to m and j runs through 1 to m you will get an M / M matrix with the ijth entry as P ij the transition probability between i and j. This is known as the transition probability matrix. So, this is also an equivalent representation of a finite state DTMC.

You can either draw the directed graph as we did the last time or you can just give the P matrix the transition probability matrix with the P ij entries. So, given one you can always recover the other it is a easy exercise to go from the transition probabilities to the directed graph and vice-versa. Now the behaviour of this Markov chain can be understood in terms of some of the properties of this matrix.

So, we will look at this matrix very closely eventually we will discuss its spectral properties like Eigen values, Eigen vectors and all that. Now, I want to talk about how to recover the n step transition probability matrix or n step transition probabilities from the P matrix. Now let us say I am just interested in considering P i j two step transition. I want to be able to go from i to j in exactly two steps.

So, can I write this in terms of the entries in the transition probability matrix is the question. So, essentially I want to go from i to some other state k and then go to j. Notice that this k itself could be i or j, but the important thing is that after two steps I have to be in j. So, as you can see this is you can just use the theorem of total probability. So, you are just looking at let us write it this way X n + 1 equals j given X n – 1 equals i.

So, this is the definition of P ij 2 so which I can just expand using the theorem of total probability into probability of going from; so X n + 1 equals j given X n equals k and X n – 1 equals i times the probability that I go from X n equals k given X n – 1 equals i. So, this is the probability that X n equals k given X n – 1 equals i and I sum over all k let us say in this case it is a finite number of states k = 1 to M.

So, this k that the intermediate state I go to could be anyone of the other states or it could be one of the states i and j themselves. The important thing is that you have to be in j after two steps of n + 1. Now the nice thing is that I can get rid of this guy does not matter because of the Markov property. So, this is by the theorem of total probability these equality is by the law of total probability.

By Markovity I can get rid of this conditioning the purples cross which leads to; so this term just becomes P X n + 1 equals j given X n equals k. I can get rid of the second conditioning on X n – 1. So, bottom line is that this becomes sum over k equals to 1 to M. So, this guy is just P kj so this is P kj P ik. And that is what the two step transition probability is. So, P ij 2 is just that.

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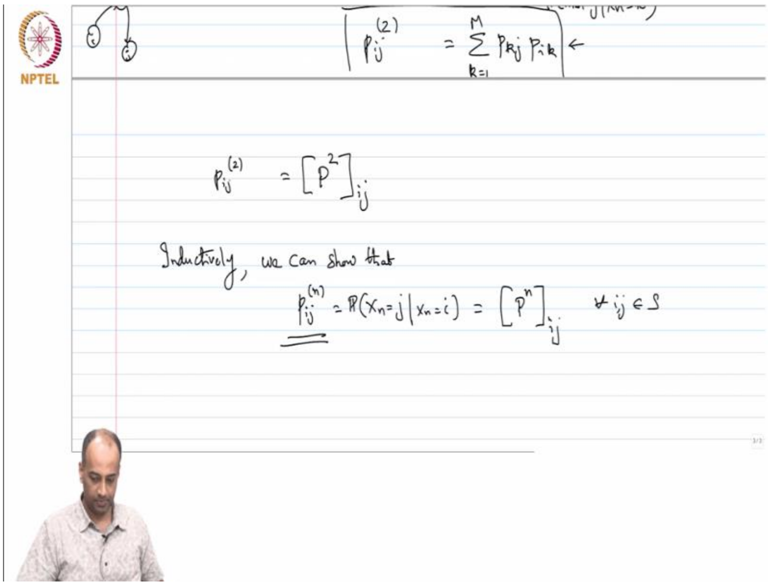

So, if you were to look at the two step transition probability matrix what really happens is this equation is just the multiplication of the matrix P with itself. You can just write it out and see. So, what we are saying is that the matrix of P ij 2 is just the ijth entry of P square. So, this is just the if you look at the matrix P square this guy on the right hand side of this equation is just the ijth entry of the matrix P square the matrix P multiplied with itself that is something about it.

So, maybe I should write it like this so the ijth entry of P ij 2 the probability of going from i to j in two steps is the ijth entry of the matrix P square. So, similarly inductively we can show that P ij n which is the probability that I go from i to j in n steps is simply the you take the nth power of the matrix P and you take it ijth entry. So, if you want to find out P ij n from the transition probability matrix P what you have to do is to take the nth power of the matrix and then look for its ijth entry.

So, P ij n so there is a common mistake that people take P ij n to be the nth power of P ij that is not correct. P ij n is the ijth entry in P power n so please keep this in mind. So, this is great. So, we can characterize P ij n the nth step transition probabilities in terms of the one step transition probabilities by simply taking the nth power of the matrix and looking for its ijth entry. This is true for all ij in S.

Here we do not need any ergodicity or any such thing. So, this is all for any finite state DTMC no further assumptions.