Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology – Madras

Lecture –45
Periodicity in a DTMC

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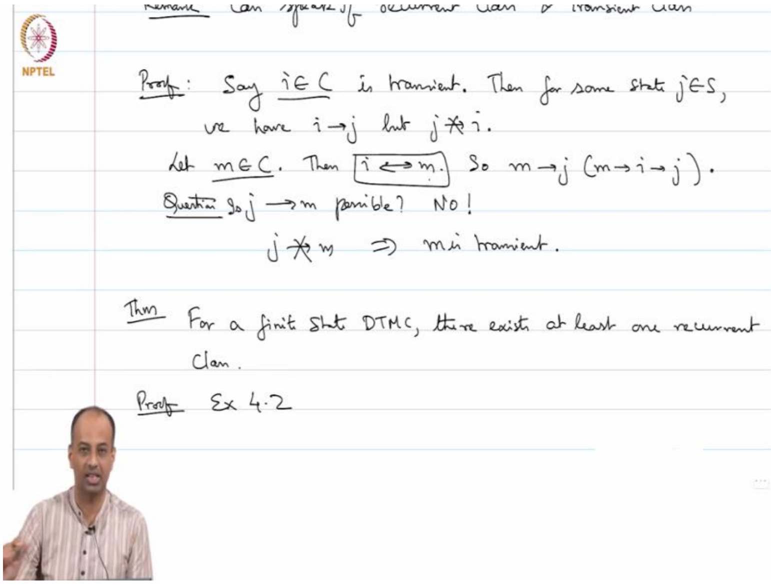

So far we have described the classification of states into recurrent and transient states. We have said that when the state belongs to these classes these classes can also be classified as entirely recurrent or entirely transient that is the summary of what we have said so far.

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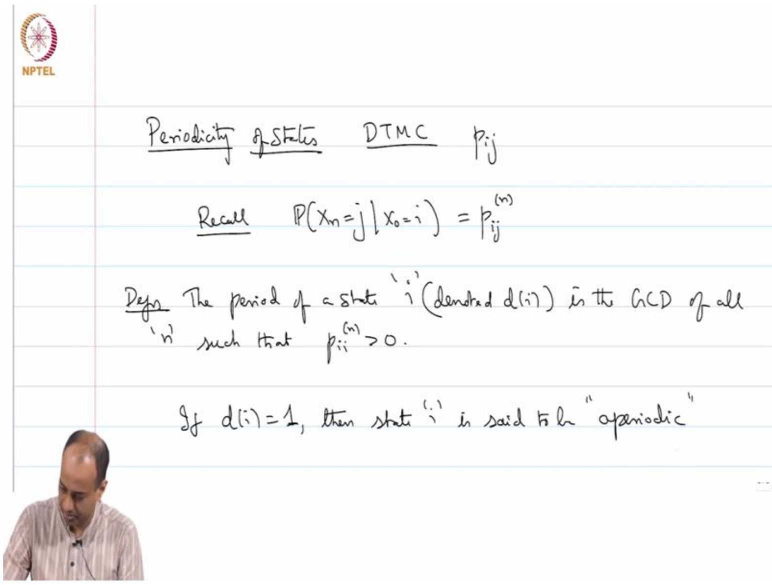

Next, we will proceed to discuss periodicity of states. This is another classification. States can be classified according to their periods. Now I have to tell you what a period of a state is.

Let us say I am given a DTMC with transition probabilities P ij recall that that n-step transition probability is defined as probability that X n equals j given X 0 equals i we denote this using P ij superscript n. Now periodicity means that in some Markov chains you will be able to go from a state i back to the state i in only integer multiples of let us say some period. So, that is what this periodicity means.

Let me define it. The period of a state i denoted d of i is the GCD greatest common divisor of all n such that P ii n is greater than 0. So, you are looking at going from state i back to state i in some n steps and you look at all those n for which P ii n the n-step transition probability is positive. You take the GCD of all those n and that is called the period of state i. If d of i is equal to 1 then state i is said to be a aperiodic.

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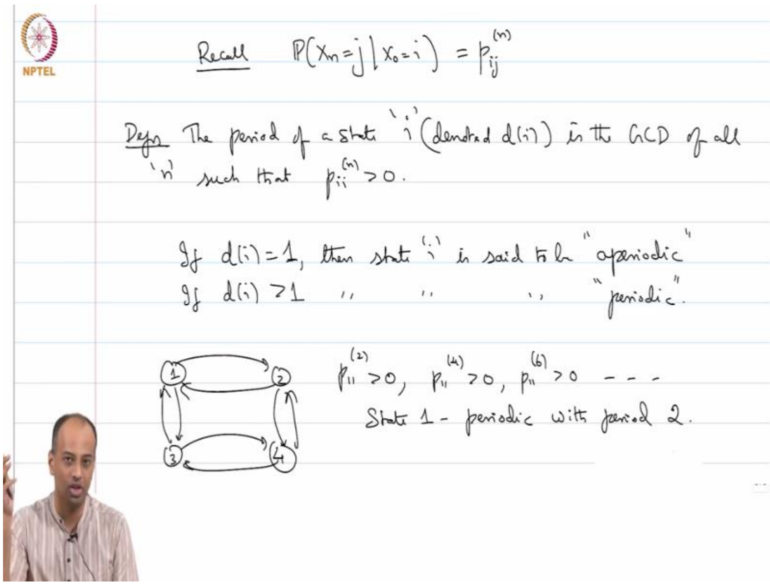

If d of i is greater than 1 then state i said to be periodic with whatever period d of i. So, for example may be just draw this Markov chain just let us say four states I have just drawn 4 states and I have just put arrows for those transitions which are possible with positive probability. So, I can go from 1 to 2 or 1 to 3 and go from 3 I can go from 1 or 4 and so on. So, as you can see so if I start at 1 so I can come back to 1 so I can go from 1 to 2 and then back 2 to 1 in two steps or I can go from 1 to 2, 2 to 4 then back from 4 to 2 and 2 to 1.

So, what you will see is that in this case for example P 11 2 is greater than 0 P 11 4 is greater than 0 and so on. P 11 6 is greater than 0 and so on. Bottom line is that you can never go from back to 1 in any odd number of steps. So, P 11 5 or P 11 3 is not strictly positive it is just 0. So, in this case state 1 is periodic with period 2. So, this is the meaning of periodicity. If it so happens that the state has period 1 which means the GCD is 1 then such a state is said to be aperiodic state.

The period is anything greater than 1 if it is 2 or 3 or whatever then the state is said to be periodic with that period.

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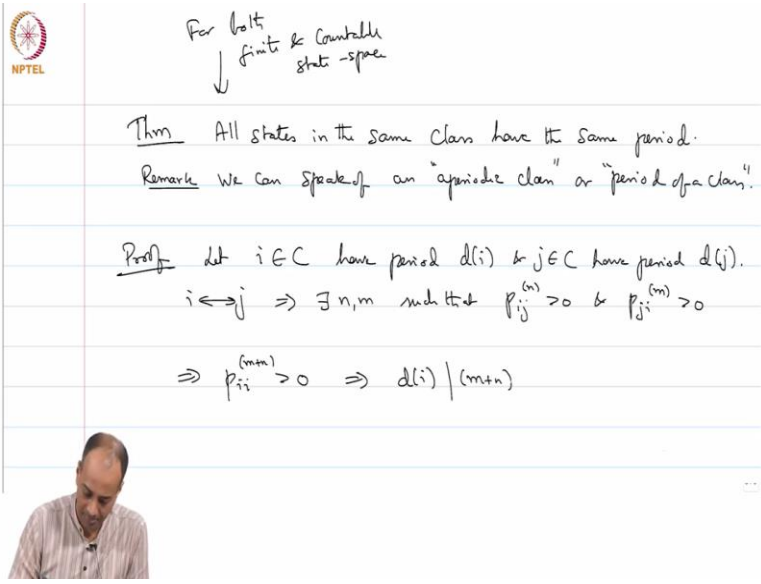

Now we state an important theorem which states that all states in the same class have the same period. So, this just means that we can speak of an aperiodic class or say things like a period of a class and so on. We can speak of the period of the class because all the states in the class will have the same period you can never have different periods and if even one state in a class is aperiodic then the entire class will be aperiodic.

So, we can speak of an aperiodic class that is what this theorem is saying. This theorem is true for finite and countable state space. The proof is a nice number theoretic sort of a proof. Let i belongs to class C have period d of i and j in the same class have period d of j. We have to prove that d of i and d of j are equal. So, now this i and j are in the same class. Therefore, i and j communicate which means there exist n and m such that P ij n is strictly positive.

And P ji m is strictly positive which means that P ii m + n must be strictly positive because I can go from i to j in n steps and j to i in m steps. So, you can from i to i in m + n steps. This implies that d of i must divide m + n. Remember that d of i is the GCD of all those case for which P i k is positive.

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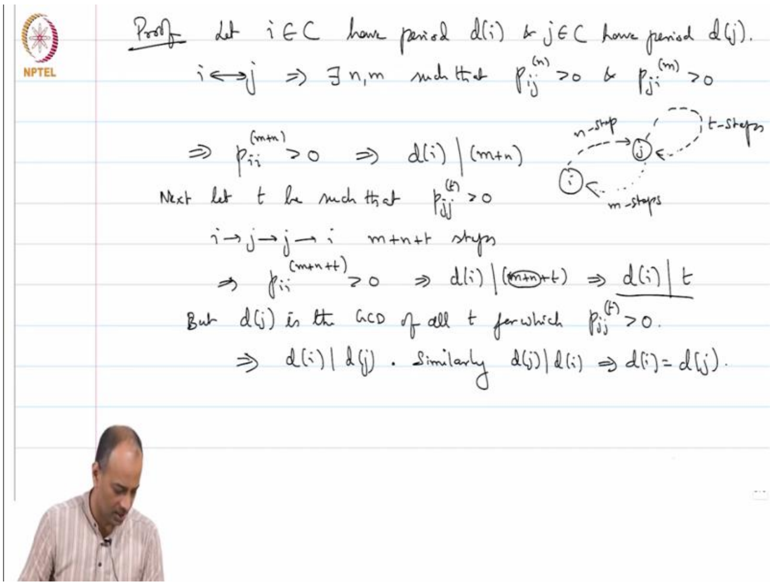

Now next let t be such that P jj t is strictly positive. Let t be any integer such that P jj t is positive. Now so what I can do so from i to j I can go in n steps and j back to i I can go in some m steps. And I am considering a t such that I can go from j back to j in t steps. Now, therefore I can go from i to j, j to j and j to i in m + n + t steps. This implies P ii of m + n + t is greater than 0.

This implies d of i divides m + n + t, but we already know that d of i divides m + n. This implies d of i divides t. So, what have we proven we have proven that d of i is a divisor of t, but d of j is the GCD of all t for which P jj t is positive. So, that is the definition of d of j, d of j is the greatest common divisor of all t for which P jj t is positive. We are saying that d of i is a divisor of any such t.

This implies that d of i must divide d of j because d of j is the greatest common divisor of any such t and d of i is a divisor of such a t. So, d of i must divide d of j. Similarly, you just reverse the roles of i and j you can prove that d of j divides d of i. This only implies that d of i equals d of j. So, this is a very combinatorial number theory type of a proof to show that all states in a class have the same period.

So, you can speak of the period of a class or you can speak of the entire class as being aperiodic or periodic. Next, I will stop by stating one theorem without proof so it is useful.

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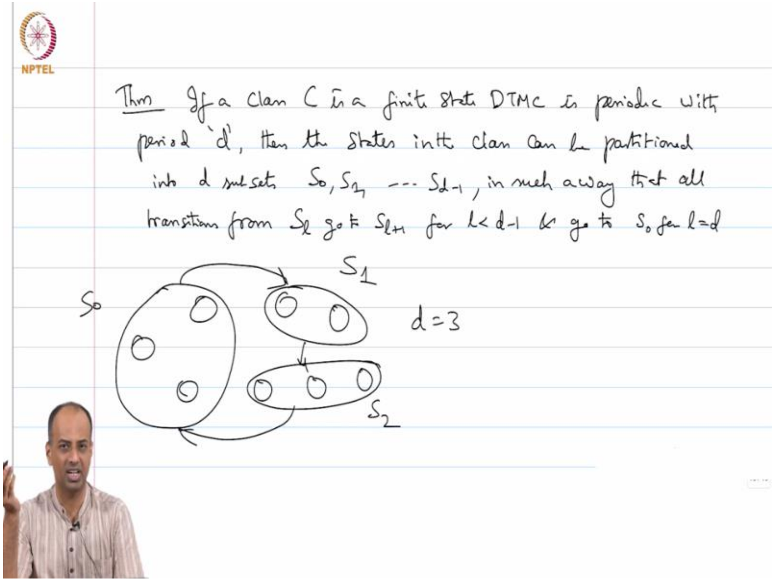

This is only true for finite state Markov chains. If a class C in a finite state DTMC is periodic with period d then states in the class can be partitioned as follows into d subsets. Let us say S 0, S 1 dot, dot, dot S d – 1 in such a way that all transitions from S l go to S l + 1 for l less than d, d – 1 and go to S 0 for l equals d. So, what we are saying is that let us say this is some periodic class.

So, what we are saying is that we can partition the states in the class there exist a partition for the states in the class. Let us say that is S 0, S 1 this is S 2. So, I have taken d equals 3. Let us say the states in a class of period this S 0 to S 1, S 1 to S 2, S 2 back to S 0. You will have transitions going from S 0 to S 1, S 1 to S 2 and S 2 to S 0 and back and if the period is d you will have d such partitions of the states in the class. So, this I will state (()) (15:51) and I will stop here.