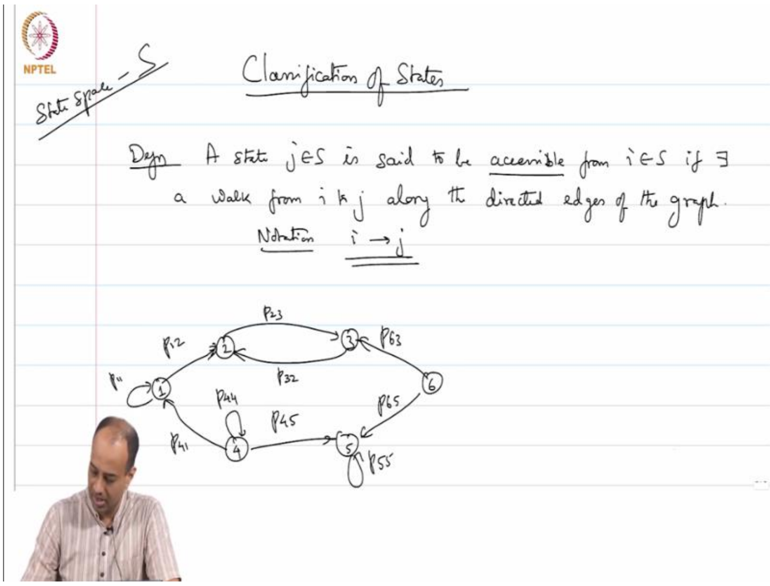
Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology – Madras

Lecture –44
Class and Types of Classes in a DTMC

(Refer Slide Time: 00:16)

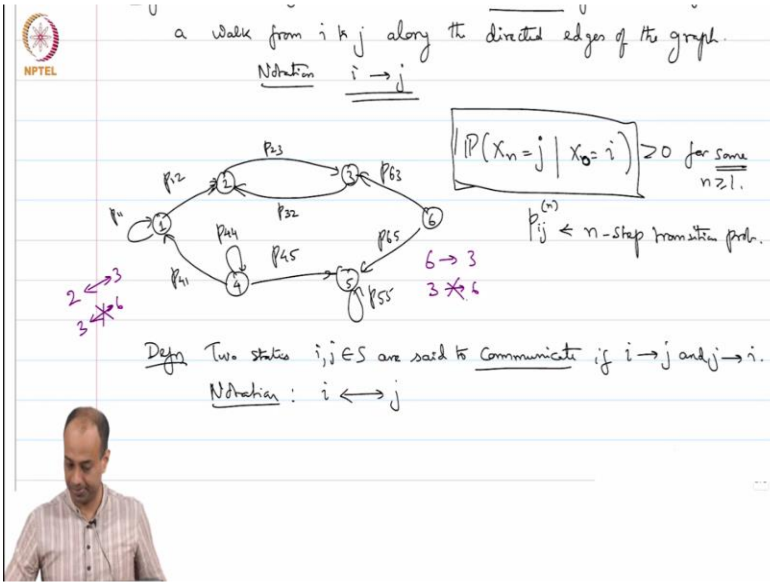


Welcome back. Now, we will discuss classification of states in a discrete time Markov chain. Now I am just going to keep this Markov chain from the previous module just for illustration purposes this is the Markov chain we had in the previous module. Definition let us say so state space is S this S that is given to you. Definition a state j in S is said to be accessible from another state i if there exist a walk from i to j along the directed edges of the graph.

So, we are given the directed graph representation of this Markov chain. We say that state j is accessible from state i if colloquially speaking if you can go from i to j along the directed edges. So, notation for this is i arrow j notation is i arrow j. So, in this definition I have not really told you what a walk is your book gives you a detail definition of what a walk is, what a path is, what a cycle is and all that.

So, walk just means that you can start at i and go to state j along the directed edges infinitely many steps.

(Refer Slide Time: 02:36)

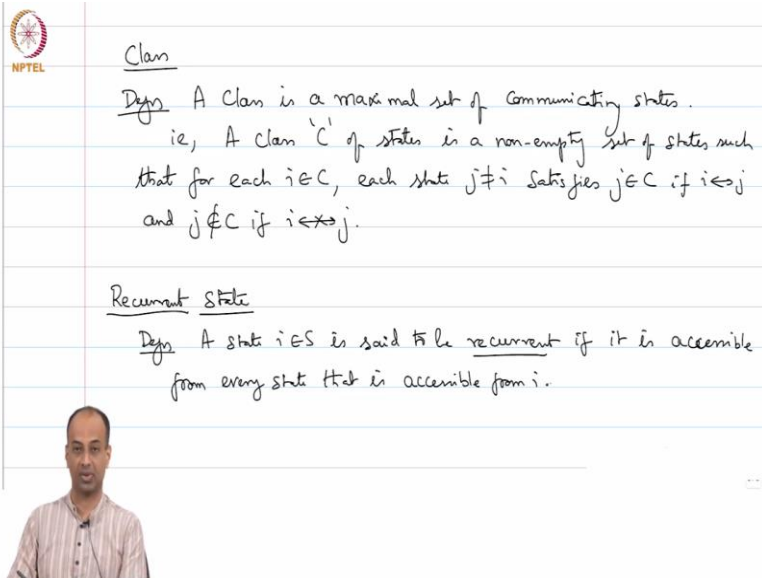


Equivalently this just means that; so i arrow j means that probability of going to state j even you started. Let us say you started at time 0 we start at i the probability that you go to j is greater than 0 for some n not all n for some n greater than or equal to 1. This just means that if you start a time there is a positive probability of going to state j in some finitely many steps.

Remember this is for some n not for all n. So, this probability this is sort of like an n step transition probability which we will study in greater detail later. So, let us just denote this P, i, j superscript n this is the n-step transition probability of going from state i to state j in exactly n steps. So, i arrow j or rather state j accessible from state i means that P, i, j n is strictly positive for some n greater than or equal to 1.

This is a very useful concept and at a related concept is the following two states i and j are said to communicate if i arrow j and j arrow i. So, if i is accessible from j and j is accessible from i then these two states i and j are said to communicate. The notation for this is i two headed arrow j. So, if you look at this picture you can write 2 double headed arrow 3. So, 3 double headed 6 is wrong it is not true because you can go from 6 to 3, but you cannot go from 3 to 6 is not possible. So, state 3 and 6 do not communicate. So, this is the definition of communicating states.

(Refer Slide Time: 05:53)



Next we define a very important notion of a class definition. A class is a maximal set of communicating states. What does that mean? So, a class is a subset of states all the states in the class should communicate between themselves and some state which is not in the class will not communicate with any state in the class that is what this means. So, let us try this a little more formally ie a class C of states is a non-empty set of states such that for each state in the class each state j not equal to i satisfies j belongs to C if i communicates with j and j is not in the class if i and j do not communicate.

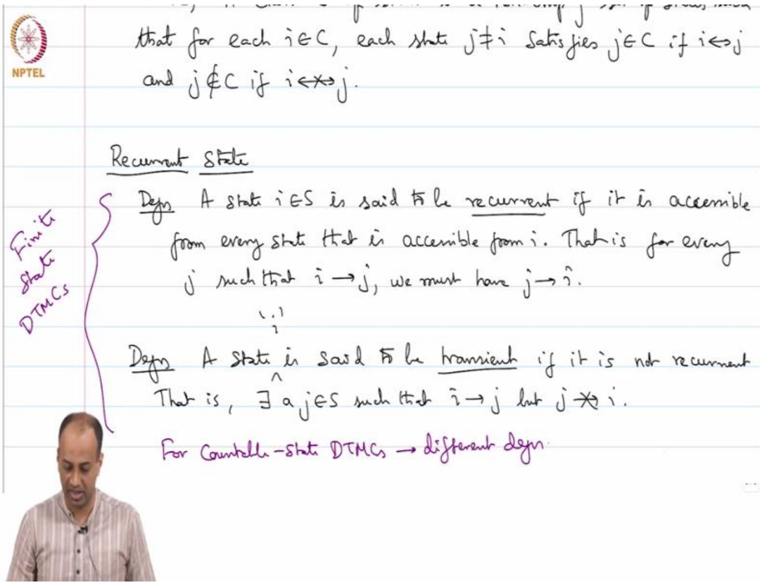
So, in this sense a class is a maximal set of communicating states. So, in a class all the states any two pair of states in the class should communicate and if you take a state which is not in a class that state should not communicate with any state in the class. So, if you go back to our picture over here this 2 and 3 form a class because 2 and 3 communicate. Now as it happens these guys these state 1, state 4 etcetera they do not communicate with any other state.

So they are all single terms. So, in this Markov chain that I have drawn out in this example the classes are 2, 3 form the class and 1, 4, 5 and 6 form separate classes by themselves I hope that is clear. So, in that class a class is a maximal set of communicating states. So, what happens is that the Markov chain the state space S get portioned into various classes in the finite state Markov chain they will be only finitely many classes.

Next, we define a very important concept called the recurrent state. Definition the state i in the state space is said to be recurrent if it is accessible from every state that is accessible from

i. So, what this means is that a state i said to be recurrent if I go from; so any state I can go to from i infinitely many step I should be able to come back to i.

(Refer Slide Time: 11:10)



In other words for every j such that i arrow j we must have j arrow i. So, for every state j I can get to from i if I can get back to i from j and if this is true for every such j then such a state i is said to be recurrent. So, even if there is one j for which I can go from i to j, but there is no way for me to go from j to i infinitely many steps then such a state i is not recurrent. So, a state which is not recurrent is known as a transient state.

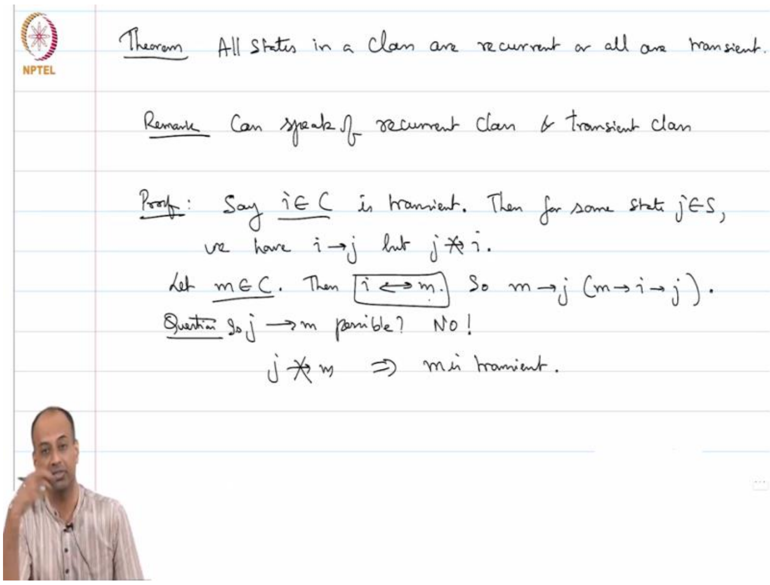
Definition state is said to be transient. Let us say state i is said to be transient if it is not recurrent that means that is there exists a j such that i arrow j, but j arrow i is not possible. So, if there exist even one j such that you can go from i to j infinitely many steps, but you cannot go from j to i then such a state i is said to be a transient state. By the way this definitions hold only for finite state DTMCs.

For countable state DTMCs different definition will apply. So, this definition of recurrent and transient states that we have given above is only for finite state DTMCs. When we study countable state DTMCs these definitions will not apply we will have to give a more involved definition later, but for finite state this is the definition of recurrent and transient states. So, if you go back to our picture state 6 here is transient.

This guy is transient why because you can go from 6 to 3, but you cannot go from 3 to 6 ever there is no way to go back from 3 to 6, but the state 2 is recurrent because the only state that you can go from 2 is 3 and from 3 you can go to 2. In this case it so happens that you can come back in one step, but if you can go from 2 to any state j infinitely many step and come back infinitely many steps for every such j then that state is recurrent.

Likewise, you can just and classify these states as recurrent or transient state 4 for example is transient because you can go from 4 to 1, but you can never come back from 1 to 4 infinitely many steps.

(Refer Slide Time: 15:48)



Now I will state a very important theorem. All states in a class are recurrent or all are transient. This is a very important theorem. This holds for finite state DTMCs as far as countable state based DTMCs. However, we will do a proof of this theorem only for finite state space DTMCs for now. So, this theorem states; so we know what a class is. A class is a maximal set of communicating states.

So, you have this Markov chain which has been partitioned into various classes. What we are saying is that every state in a class is of the same type. It is either all the states in a class are recurrent or all the states in a class are transient. You cannot have some recurrent states and some transient states within the same class not possible that is what this theorem is saying. So, remark what we are saying is that we can speak of the recurrent class and transient class.

So, the class is either entirely recurrent or entirely transient. Proof and this is only for finite state Markov chains. So, let us say that class is say is i belongs to a certain class is transient. We want to show that if there is one transient state in the class then all other states in the class will be transient. Then for some state j in the entire state space we must have i arrow j is possible, but j arrow i is not possible.

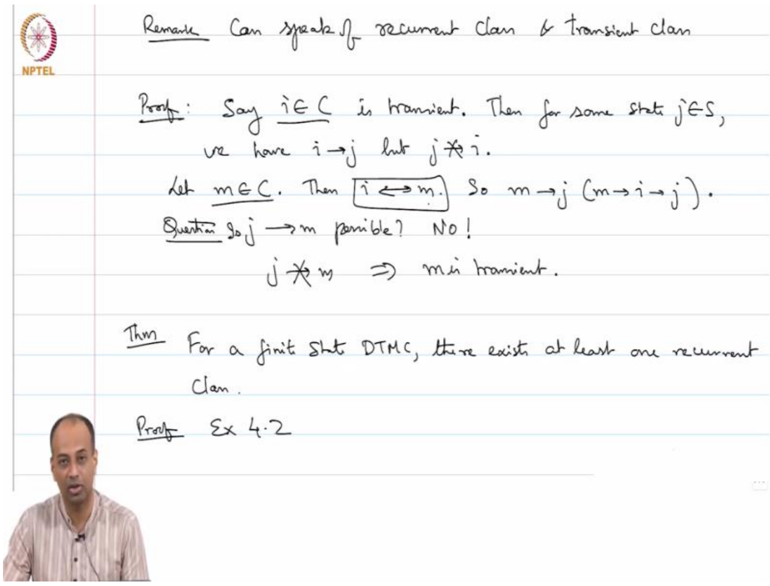
This is the definition of a transient state i is transient we are taking i to be transient so there must exist a state j which could be anywhere in the state space not necessarily in the class such that i to j, but j to i is not possible i to j is possible, but j to i is not possible. Now let m be any other state in the class then by definition of class we must have i double arrow m this is a definition of class.

So, m arrow j why that is because I can go from m to i then I can go from i to j. Now the question is j arrow m possible? Can you go from j to m in finitely many state? The answer is no because if you did I can go from j to m and anyway I can go from m to i because i and m communicate which means I will be able to go from j to i through m. So j to m is not possible.

So, m arrow j, but j arrow m is not possible. This implies m is transient. So, we have shown that if you have some transient state i in a class than any other state m in the class must necessarily be transient. So, in other words if you have even a single state in a class which is transient then every state in the class must be transient. So, which means that in a class you cannot have a mixture of recurrent and transient states because the moment you have a transient state the entire class becomes transient.

Of course, all states in the class can be recurrent. What we are saying is that we cannot have a mixture and we have proved it. So, you can speak of the entire class as being a recurrent class or a transient class because the states in the class will all be of the same type recurrent or transient this is a very important result. This proof only holds for finite state DTMCs. Although, the theorem itself holds for even countable state space DTMC which we will prove later, but this proof is only for finite state DTMCs.

(Refer Slide Time: 22:12)



Another interesting result which I will not proof is the following. For a finite state DTMC there exists at least one recurrent class. This is a very nice important result. If you have a finite state DTMC as I said earlier in any Markov chain the states are partitioned into classes. These classes can be either each of these classes could be a recurrent class or a transient class. What we are saying is that in finite state DTMCs we are guarantee to have at least one recurrent class.

So, there may be more recurrent classes, but we guarantee at least 1 and transient classes they may or may not be you are not guaranteed anything. So, this proof I will not go into. It is a fairly simple proof I think in Gallagher exercise 4.2 there is a guided proof of this. It is simple enough, but it is good to keep in mind that in any finite state Markov chain we have guaranteed at least one recurrent class.