Stochastic Modeling and the Theory of Queues
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Lecture –41
Ensemble Rewards: Age and Duration Continued

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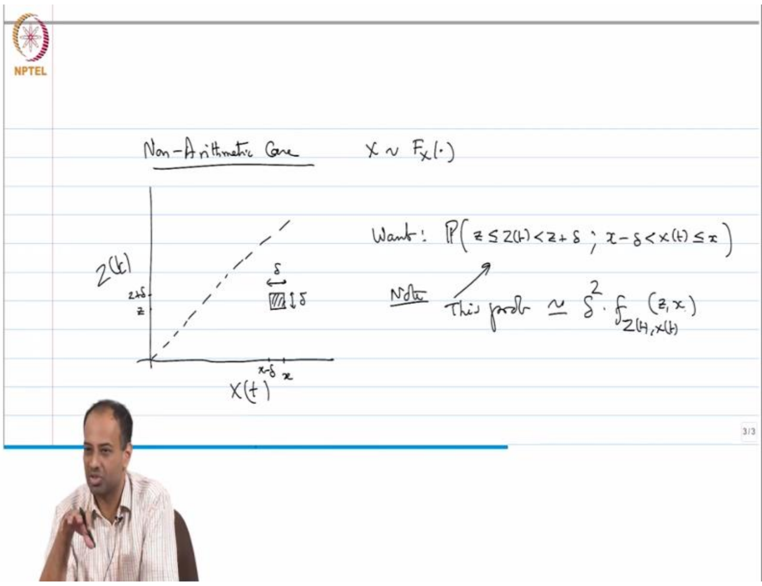

Now let us proceed to the non-arithmetic case if you do not have any questions in the arithmetic case. The ideas are essentially the same, but now in the non-arithmetic case you know these age and duration they are not restricted to integer values or any such thing. So, you have to look at some small interval delta and look at what is the probability that Z t lies in z – delta to z and X t lies in x – delta to x and all that.

So, here I am going to take x is distributed according to some F x this is not arithmetic this could be some generic cdf. Now I want to plot X t against Z t. Of course, X t is so where will this joint distribution lie? It will lie above this dotted line or below this dotted line. Below this dotted line because X t be smaller than Z t. Now what I am going to do is I am going to take some particular x and some particular z.

So x z there is some point here I am going to take a little square which lies in so this is a delta square. I am going to look at the probability that X t Z t lies in this little square. So, I want to calculate the probability that Z t we should write it like this little z less than or equal to z t less than z + delta and x – delta less than x t less than or equal to x. So, maybe I should draw it this z to z + delta.

So, my little z is there z + delta is here and this is x – delta so I want to look at the probability that at some particular time t my z t between z and z + delta and x t is between x – delta and x. Suppose, I get this answer so what is this roughly like? Intuitively speaking let us say that so just for the sake of argument let us say that x is a continuous random variable with density little f x.

None of that is necessary for this, but let us say x is continuous which means that X t and Z t will also be some continuous jointly continuous random variables. Suppose, I manage to calculate this probability that I have written out here that X t and Z t lie in the little square. What will that approximately be like? It will be delta square times the joint pdf of Z t and X t that is what it is.

So, this will be like this probability is approximately like delta square times joint density assuming that it exists for small delta. I am just trying to say what this mean? You know if you take a joint density let us say you take forget age, duration and all that let us say you take jointly continuously random variables x and z it will have some joint pdf. The joint pdf does not have any interpretation of probability.

If you multiply it by a small area if you take some little x little z and you put a small area around it the small area times the density is the probability that your random variables lie in that little regions so that is the interpretation of joint density that is what I am saying. So, if I calculate it is as good as calculating a joint density. Of course, there is no guarantee that the joint density will exist.

But I am saying if it did this is the interpretation. So, if I calculate this I have a handle on the joint distribution.

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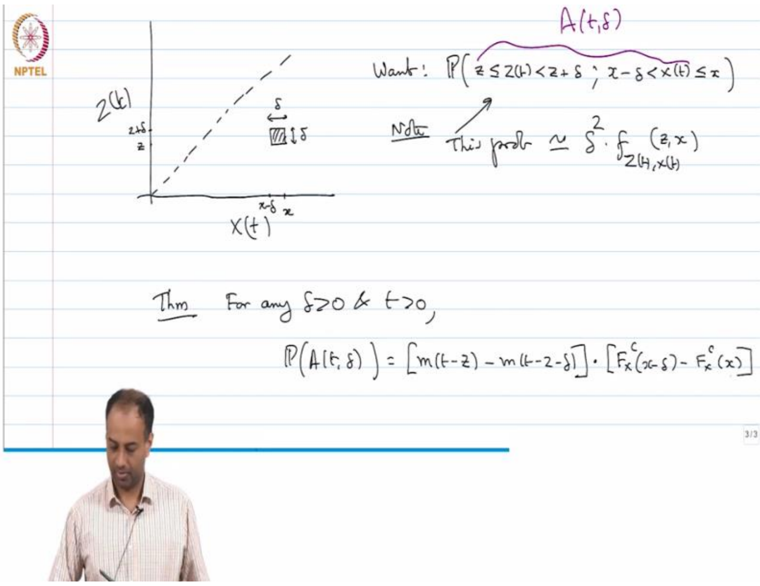

This is not a very rigorous statement I am saying it is approximately like delta square times the joint density. So, I am going to call this event as A this is event A. Theorem probability for any delta greater than 0 and maybe I should call this A of t, delta. This is for t delta at z. So, for any delta greater than 0 and t greater than 0 probability of A, t delta is equal to m of t – z so let me write here I would not have enough space let me write it like this.

Probability of A, t delta = m of t – z – m of t – z – delta times F x compliment of x – delta – F x compliment of x.

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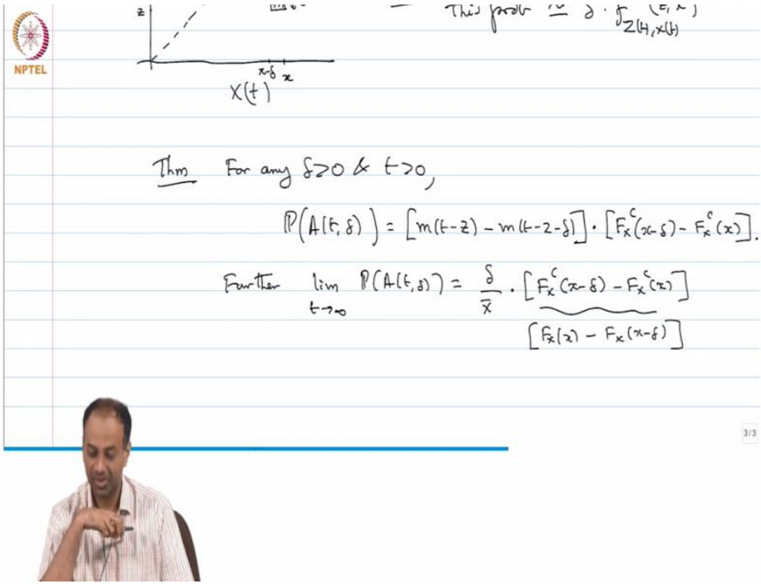

What is m of t – z? It is expectation of n of t – z m of t we already know what it is. So, this is what it is I have a characterization of this probability that I lie in this small square around z and t and also further limit t tends to infinity probability of A, t, delta is equal to what do you expect? You should tell me this so I am telling you an answer in the first part and I am sending t to infinity.

So, the first if you look at the expression for probability A, t, delta it has an m of t term and F x term. This F x term has no t dependence so it will just remain what it is and what is the limit t tending to infinity of m t – z term m t z – m t z – delta. It is the expected number of renewals in t – z – delta – t – z as t tends to infinity this should go to delta over X bar by what theorem Blackwell's theorem.

This is a non-arithmetic renewal process. This is true for any delta greater than 0 delta does not have to be small actually this is true for Blackwell whole square any delta. Delta over X bar times the rest is the same or actually this bit can also be written as F x of x – F x of x – delta that is also okay. This bit is the same as these two square brackets are equal either write it in terms of complementary cdf or the cdf it does not matter because the F x compliment is 1 – F x so you can get this.

So this is great so this is the answer. So, now if delta were small so I think before you prove this I will indicate how to proof this in next 10 minutes, but this big F x corresponds to some continuous random variable it will have some density little f x. Suppose x has a density little f x then what will this be equal to? The first term is delta over X bar the second term will be like so if this is the probability that x lies in x – delta to x which is simply delta times little f x.

So, finally you will get something like delta square F x of x over X bar which means the joint density is like F x of x over X bar for 0 less than or equal to z less than x.

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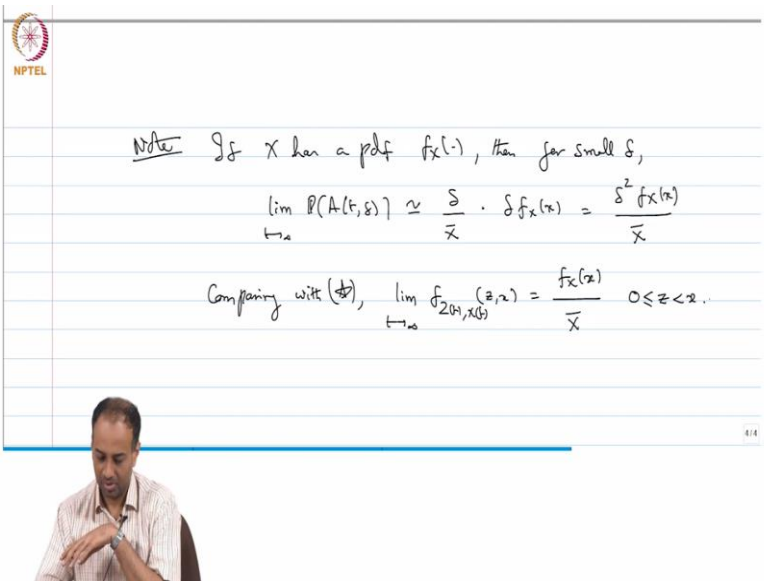

So, note if X has a pdf little f x then for small delta limit t tending to infinity probability of A, t delta will be like delta over X bar times delta f x of x which is nothing, but delta square f x of x over X bar. So, if you take this as sum star which basically you are looking at probability of A t delta for large t is roughly like delta square times the joint density. So, what we are saying is that so we can write limit t tending to infinity of z, x is like delta square, delta square will cancel.

So, you will get over X bar for 0 less than or equal to z less than x. So, if you had a joint density if x has a density if Z t also has a density. Thus, you have asymptotic joint density of z and x is given by this. From this I can calculate the marginal of Z t marginal of X t expected (()) (14:27) t and all that. So, this is for small delta, but the result I have stated is true for any delta greater than 0 or any t greater than 0. So, this probability A t delta.

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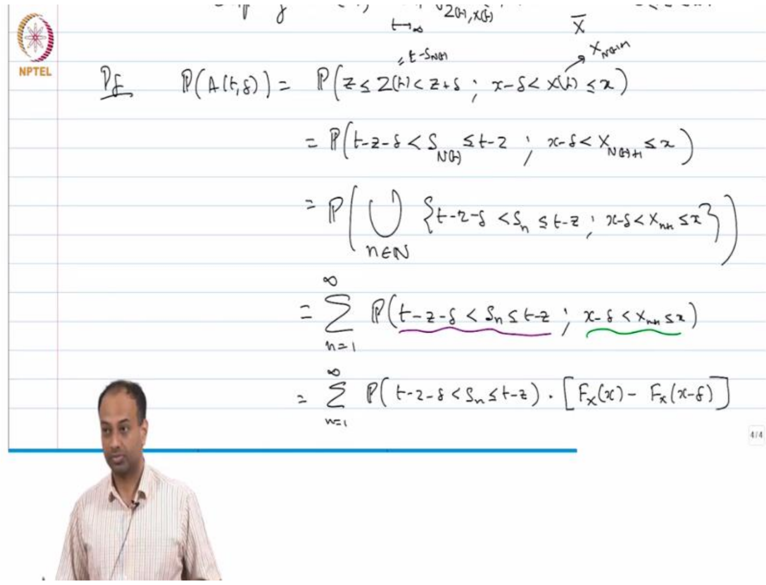

So proof I will just gets the proof. Probability of A, t, delta is just probability that little z less than or equal to Z t less than z + delta and x – delta less than x t less than or equal to x. This I can write I want to write this X t Z t in terms of S N t S N t + 1 which I already know. Z t is nothing, but t – S N t and X t is X N t + 1 this S N t + 1 – S N t. So, this I can write it as probability that t – z – delta less than S N t this is just algebraic manipulation this you can check t – z and x – delta less than X N t + 1 less than or equal to x.

What have I done? I have just written X t and Z t in terms of S N t + 1 and S N t. I hope this is not very mysterious I am just writing so this Z t is what? This Z t is nothing, but t – S N t and likewise this guy is nothing X N t + 1 that is all that I have done. I have just rearranged things. So, I am looking at this probability that S N t the epoch just before t occurred between t – z – delta and t – z which is how your age is to be between z – delta and z or Z N z equal to delta and unlike for duration.

Now N t this now you can write this in terms of union of this joint events N t can take value 1, 2, 3 etcetera. So, I can write this as probability of union n belongs to N. The same thing t – z – delta less than S little n less than or equal to t – z and x – delta less than X n + 1 less than or equal to x. So, basically the event above this can happen in one of this joint ways where n can take value 1 or n can take value 2 and so on.

You can show that this union is equal to the previous event and you can also show that this union is the disjoint union. So, if you have this disjoint union countable union you can write this as sum that is the main thing sum n = 1 to infinity probability that this is the standard trick N t has to take 1 of this disjoint value little n. So, Inside the summation I have a probability of some A intersection B type of a thing A n intersection B n, but this event so if I take that bit that is independent of that bit.

Why? X n + 1 is independent of S n so what does this become? So, this will essentially become sum over n = 1 to infinity probability that t – z – delta less than S n less than or equal to t – z times the probability that so what is this? Probability that X n + 1 lies in X – delta 2 x this I can write in terms of I can write this as F x of this is the green thing so this is this. I have used independence and this bit has the second term has nothing to do with n.

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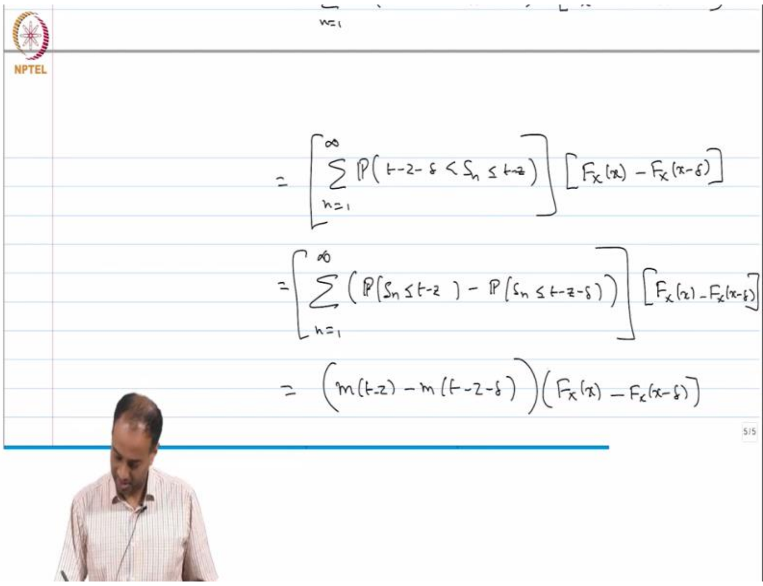

So this becomes F x this is the term I want anyway. So, I can write this like this n = 1 to infinity probability of t – z – delta less than S n less than or equal to t – z all that times F x of x – all of this is true for all t greater than 0 and all delta greater than 0. So, what is this? So, this is the most interesting part of the proof. So, this is to further to make this even clearer I can write this as sum n equals 1 to infinity.

I can just write this as probability S n less than or equal to t – z – delta – probability that S n sorry t – z is it correct? I think I did it right in the other way okay let me just get back to you so let me just write this, this is the problem. So, have I interchanged what is this and this is the question which is a bigger probability? First one is bigger so I think this is correct. So, I am just using again I am using disjoint unions.

So, what is sum n = 1 to infinity probability S n less than or equal to t – z m of t – z. So, this is nothing, but so you back to your characterization of m t, m t was characterized as sum over n = 1 to infinity probability S n less than equal to t. So, this is nothing, but m of t – z – likewise m of t – z – delta times F x of x – F x of – delta finished. Now you send t to infinity the second term will remain the same the first term will become delta over X bar.

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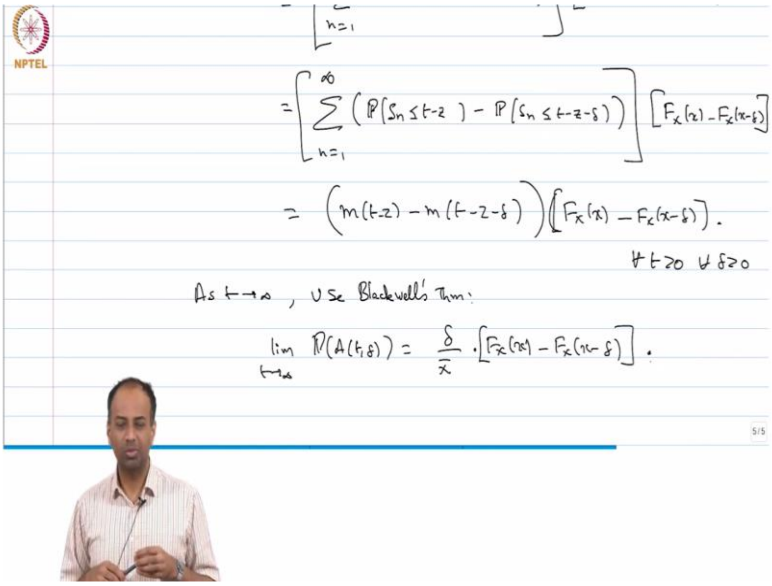

So, this is true for all t greater than 0 and for delta greater than 0 as t goes to infinity use Blackwell. So, you will write limit t tends to infinity probability of A, t, delta = delta over X bar which is the first term times F x of x – F x of x – delta finished. So, this gives you the probability that this age and duration lie in that small square. From this you can calculate joint cdf, joint pdf, marginal pdf, marginal cdf anything you want and once you know the joint distribution you can calculate the expected actually you can calculate the expectation of any reward which is a function of Z t and X t as t tends to infinity.

So, we will deal with more general rewards in tomorrow's lecture and that will be the end of this chapter.