Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology – Madras

Lecture –40
Ensemble Rewards: Age and Duration

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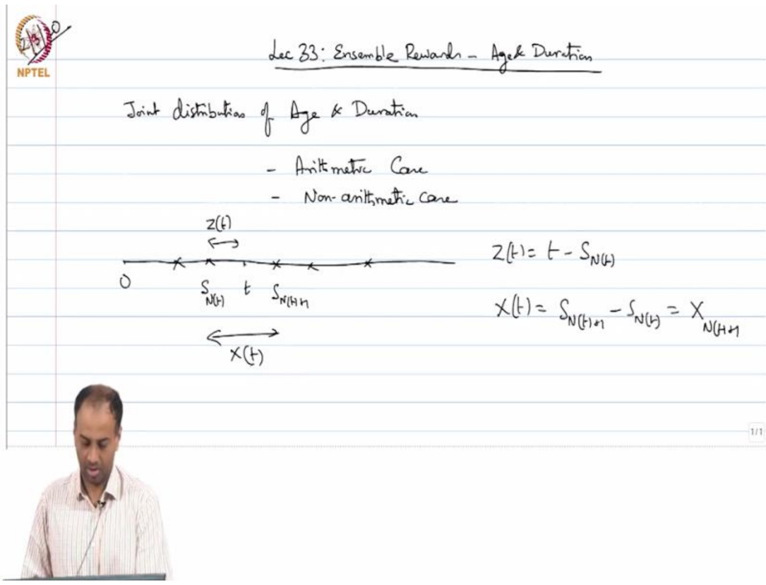

Good morning welcome back. So, the topic that remains in this chapter on renewal processes is the Ensemble reward characterization. We already did time average reward, we have to look at ensemble average reward. Before we deal with general ensemble average rewards I want to deal with the simple cases of age and duration which constitute very simple examples of reward process.

So, in this lecture we will derive the joint distribution of age and duration and we will do it for arithmetic case and non-arithmetic case separately. Now recall that if we have a renewal process and you fix some time t the epoch of the previous arrival is S N t and the epoch of the arrival after t is S N t + 1 and the age is t – N t, Z t is t – S N t and the duration X of t is just S N t + 1 - S N t which is nothing, but X N t + 1.

This we already know this is a definition of age and duration that is your duration. We want the joint distribution of age and duration. So, remember that we did time average of age and time average of duration in the earlier and I also gave homework problem saying using time average figure out the joint distribution of age and duration. Actually, it turns out you can calculate the joint distribution of age and duration for even finite t.

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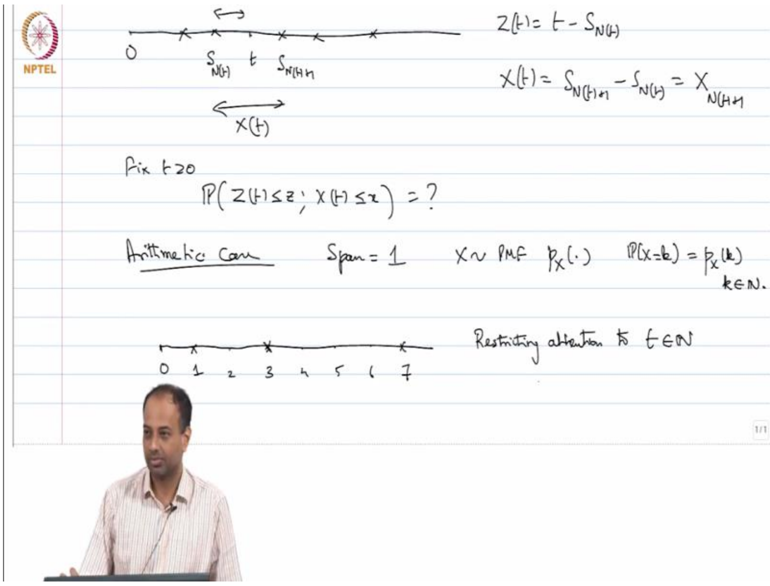

So, for any t you can fix a t and ask what is the probability that Z t is less than or equal to z and X t less than equal to x. This should be the joint distribution and this will be a function not only of little z and little x this will also depend on t for any finite t and presumably as t tends to infinity you will have some steady state ensemble reward this can be written as expectation of indicator that Z of t less than or equal to z and X of t less than equal to x presumably it will have some steady state limit as t tends to infinity.

So, we will see how it works out that the topic of this lecture we will treat the arithmetic case first because arithmetic case is it is very discrete and very easy conceptually little easier. So, this is what we want. The arithmetic case let us say span = 1 no loss of generality. What is an arithmetic process? Each renewal occurs only at multiplies of some kappa. So, in that case the probability of having a renewal in a small interval Blackwell's theorem we did already.

So, whether you have a renewal in a small interval or not depends on whether that interval includes one of these multiples of kappa the span. So, this is in some sense we can say that arithmetic renewal process in non-arithmetic renewal process behave very differently. So, we will deal with the arithmetic case first. We are taking span = 1 you can take kappa, but there is nothing to be gained.

So, this means that renewals occur at multiples of 1 which is integers the only positive integers. Of course, we are not saying that every integer there will be a renewal. We are saying that if a renewal occurs it will occur at an integer time that is all that we are saying. So, we are going to take X is distributed according to PMF P x. So, what I mean is that probability that x = k is p x of k where k is a natural number.

This is the inter renewal distribution. Now for this arithmetic renewal process we will look at age and duration taking only we can look at only just integer values because the whole process sort of becomes discrete time process. There will be you know these you can naturally sort of forget about continuous times. Everything in the process happens at the embedded integer points.

So, here is a picture this is 0, this could be 1 renewal 2 may not be a renewal, 3 maybe a renewal, 7 maybe renewal and so on. I put a cross when the renewal happens and nothing when put a little dot when there is no renewal. So, in the notation that we got so can the age be 0? Age can be 0, for example, the age at 3 is 0, but can the duration be 0? Duration is at least 1 because you are not allowing for I mean the renewals are at least separate will be 1 there is no PMF at 0 the mass at 0 is 0.

So, duration can take values 1 to 3 and age can take values 0, 1, 2 and what if I tell you that the age is sum number let us say 2. What can you say about the duration? So if I would say for example at the time 5 the age is 2 so what can be the duration values will be 3 or bigger. So, that is where the joint distribution leaves so to speak. So, I am only looking at integer times t.

So, restricting to attention to integer times t is also an N. Now so age so what we are saying is that duration is at least one bigger than the age.

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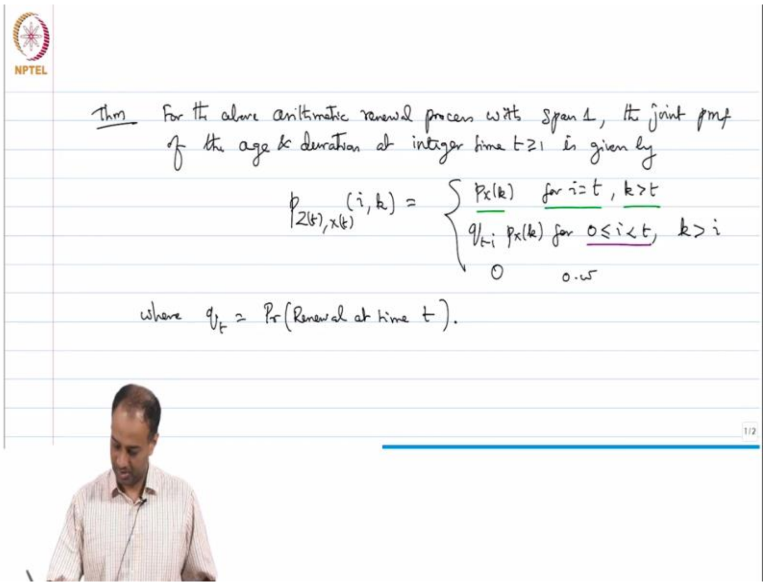

So, now I will tell you theorem for the above arithmetic renewal process with span 1. The joint pmf of the age and duration that integer time t is given by P z t x t of i, k is equal to P x of k for i = t and k greater than t q t – i P x of k for 0 less than or equal to i less than t or k greater than i 0 otherwise. So, I am looking at the joint pmf of age and duration at integer times t and little i is the variable corresponding to the age and little k is the variable corresponding to the duration.

We are saying that this is the joint distribution. Now I have to tell you what is this q t – i? q t is the probability of renewal at time t. So q t – i is the probability of renewal at t – i; so I can say where. Now so this has essentially two parts this joint pmf the first that guy P x of k for i = t and k greater than t that would deals with so when i = t what does that mean? i is the variable corresponding to age.

What does it mean to say the age is t yeah that there has been at time t the age is t that means that even the first renewal has not happened so far. So, if there has been no renewals so far then what is the probability that my duration is equal to k. It is a probability that the first renewal occurs at k for which the answer is P x of k. Suppose they tell you there has been no renewals what is the probability that my renewal will occur at k.

My duration is k it is the same as the probability that the renewal will occur at k which is P x of k which is the same as the value that the first renewal instance takes value k. So, the first part of this is fairly easy to see. The second part involving this q t – i P x of k corresponds to there are already being some renewals. So, you are looking at this part now 0 less than or equal to i less than t.

So, now the age is strictly less than t which means that there has been at least one renewal and so you are looking at age is equal to if you go back to this picture you are sitting somewhere here let us say; let us say you are sitting at this point and you are looking at this your t in this picture is 6, but you are looking at the probability that the age is i which means that there should have been a renewal at time t – i.

And what else you want the duration to be k. So, q t – i is the unconditional probability that there is a renewal at t – i and given that there is a renewal at t – i the probability that the next renewal interval will last for duration k is simply P x of k because they are independent so you get this bit, the second bit that is how this theorem comes about. Now can we characterize this q t – i for finite t it is some function of t.

But as t tends to infinity you should be able to characterize q t correct. So, this theorem is valid as it is it is exact and it is not difficult to prove.

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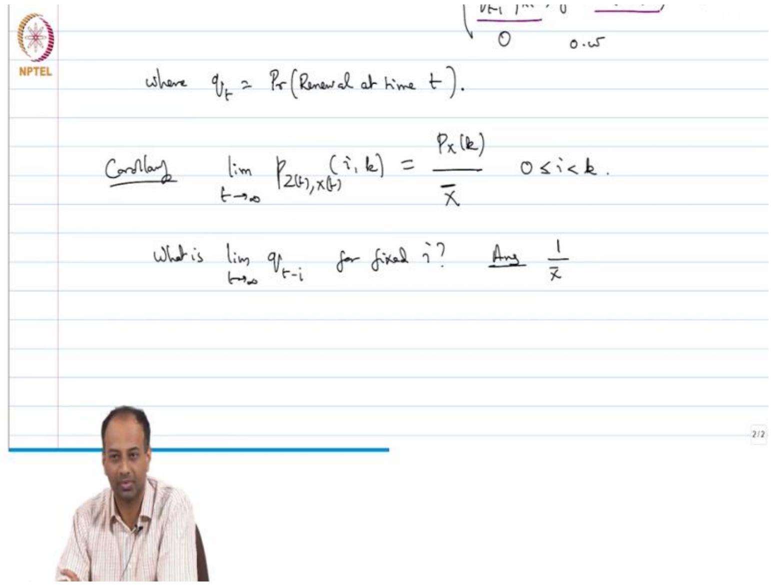

As a corollary we can say limit t tends to infinity P z t x t of i, k is equal to P x of k over X bar for 0 less than or equal to i less than k. So, this is a corollary that comes from the above result. So, essentially I want this right what is limit t tends to infinity if you look at the first term P x of k. So, when i = t the k is greater than t. So, when t tends to infinity k also tends to infinity. I have two terms to deal with in the theorem.

The first is the term in green; the second is the term in purple that guy. So, I am saying that the green term will go to 0 as t tends to infinity and the purple term is the one that remains. What happens to the green terms? So, you are taking t to infinity which means i goes to infinity for the first term in green and k is greater than t so k also goes to infinity and what happens to P x of k as k goes to infinity it has to go to 0 because of the pmf.

So, the first part goes to 0 what happens to the second part? You will have limit t tending to infinity q t i times P x k. Now what is limit t tends to infinity q t – i for i fixed. It is the probability that there is a renewal at some t – i. So, what is the probably that for a arithmetic renewal process what is the probability that there is a renewal at a particular t as t tends to infinity what is the theorem governing this?

See the expected number of renewals in a interval is equal to delta over x bar for Blackwell's theorem, but that is for non-arithmetic processes. For arithmetic processes at the multiple of span the probability of having a renewal at that multiple of a span is equal to span over X bar I think we stated this you can go back to Blackwell. So, this will simply be answer to this will be span over X bar one is the span so you get what you want P x of k over X bar.

So, this is the steady state joint distribution of duration and age. It is constant in i as you can see, but i has to go between 0 and k – 1. So, from this you can easily find the marginal of age and marginal of duration by marginalizing one of the random variable. So, actually this was a very simple elementary probability argument. The only non-trivial thing we have used in this actually the first the theorem itself as I have stated it is a completely elementary discrete probability argument involving almost nothing that is nothing that advance.

The only sort of advance thing we have used is Blackwell's theorem in taking the limit in t tending to infinity.