**Stochastic Modeling and the Theory of Queues**
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**Lecture –32**
**Elementary Renewal Theorem**

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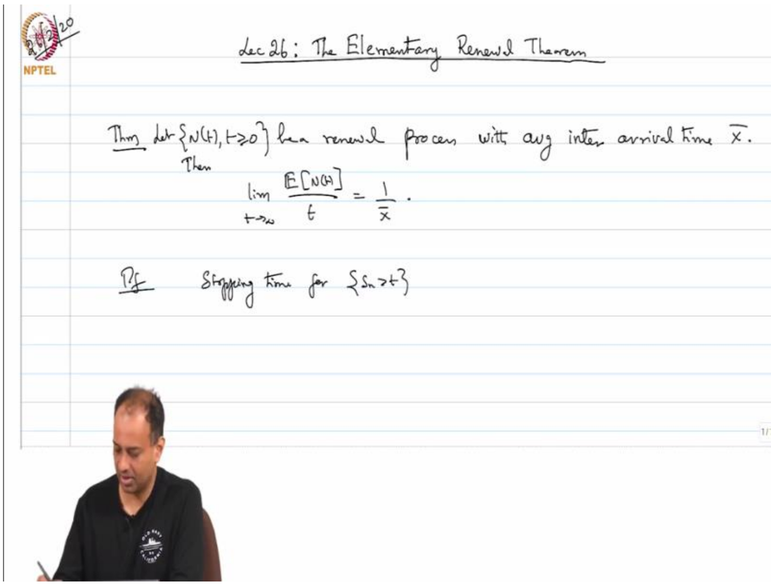


Good morning. Welcome back. Last class we proved the Wald's equality which relates the expected value of a stopping time to the expected (()) (00:28) at stopping. We will now hit towards proving the elementary renewal theorem just to refresh your memory I have stated this somewhat informally before. The elementary renewal theorem says that let N t be a renewal process with average inter arrival time x bar.

This x bar could be finite or infinite then limit t tends to infinity expected N t over t is 1 over x bar. This is the result that we want to prove, proving this involves stopping time and Wald's equality in a crucial way. We have already said that this elementary renewal theorem is not a direct consequence in any way of the strong law. N t over t of course almost really converges to 1 over x bar.

But if you take expectation there is no guarantee that it will converge to 1 over x bar I mean MCT and DCT do not apply, you cannot apply monotone convergence or dominated convergence, but there is a route through Wald's equality. Proof so we need a stopping time for S n. This is what we will first define.

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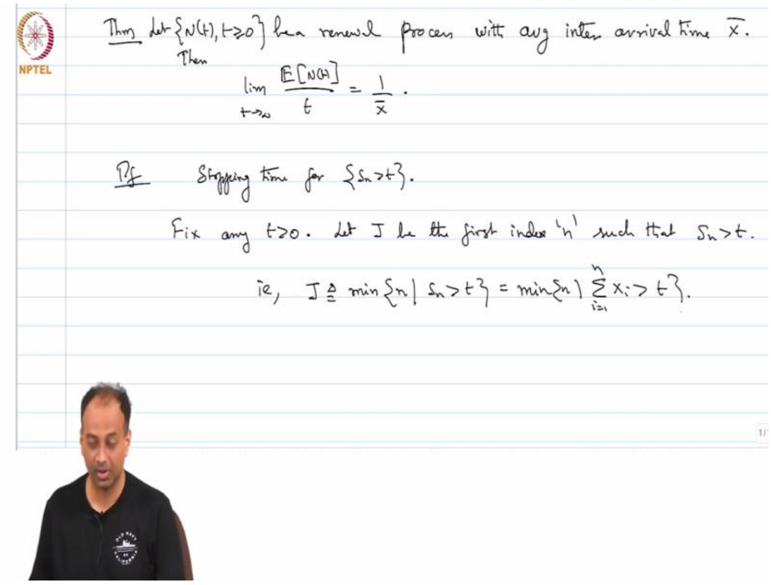


So, you are basically getting this realization of xi's. When your renewal process evolves x 1, x 2, x 3 etcetera realize. Now, you fix a t so this is the first stopping time for S n greater than t fix any t greater than 0. Let J be the first index n such that S n is bigger than t. So, what do I really mean? J is defined as the smallest n such that S n has strictly exceeded t. So, my random variable x 1, x 2 etcetera are realizing from which I make these partial sum S n.

Each time a new x i realizes I will form the partial sum. So, I am fixing t ahead of time okay you fix whatever t want finite t greater than 0 and ask what is the first index n for which my S n strictly exceeds t and that is my definition of J. So, I can just write this as minimum n such that sum over i equal 1 through n x i greater than t. So, this J functions like a stopping rule for these x i. So, for each n I will add up the first n x i and t is anyway fixed.

I will ask the question this is my sum over x i greater than t the answer will be no, no, no and suddenly I yes. The moment I say yes I call that n as J that is my stopping rule. I am not looking ahead I am just looking at whatever x i is realized I look till then and see if my sum exceed t if not I continue if it does exceed t I stop. So, this is a value stopping rule. In particular, you can see that the event this J greater than or equal to n is independent of x n + 1, x n + 2 etcetera.

So, this is my stopping rule. Now, I want to express at this point it is not even clear how I am going to get to anywhere in the renewal theorem, but just hold on for a minute.

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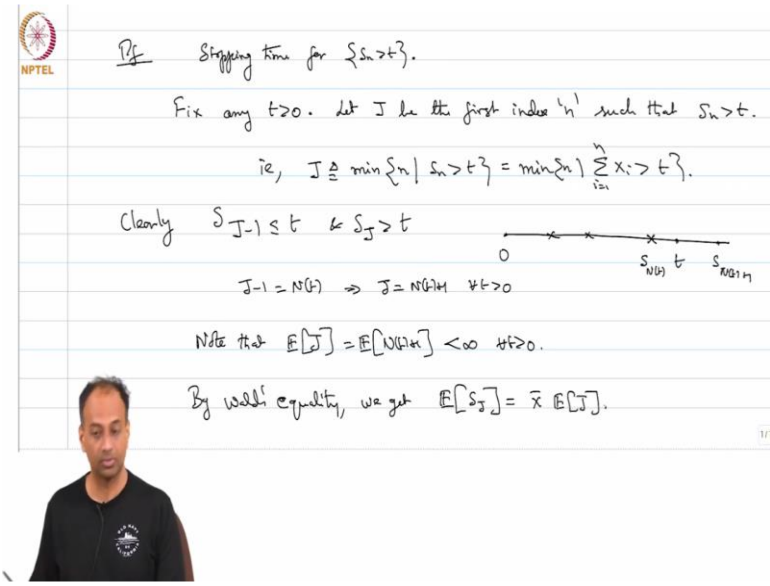


I am just going to see what this J is in terms of n of t. So, what do we have? So, I have this J – 1, J – 2 are all such that this is hold this is true for S J – 1, S J – 2 etcetera that will be less than or equal to t, but S J should be greater than t. This is just from the definition of J till J – 1 that should be less than or equal to t S J – 1 should be less than or equal to t, S J should be bigger than t then you stop.

Now, this should give you a sense of what J ought to be in terms of N of t. Let us just draw this picture a bunch of renewals running this is 0. At some time t what is the arrival before t S N t. This we know from before. There have been N of t arrivals at time t so the previous arrival was epoch was S n t and the next arrival epoch was S N t + 1 this we already know. So, in terms of S J what did we say?

S N t is basically S J – 1 and S N t + 1 S J. So, what are we saying? So, we are basically saying that from the picture it clear that J is N t + 1 from the picture it is clear. So, the first index that I know exceeds t is the index N t + 1 for every t. So, the stopping rule J is nothing, but N t + 1 it is very clear from the picture. So N t + 1 is the stopping rule for the first index that exceeds t N t is not a stopping rule that is the key issue because at N t if I just look at the random variable x 1, x 2 etcetera till x n t which leads to S N t.

I have no knowledge of whether there will be an arrival before t or after t I cannot say whether at this point as S N t epoch I will not be able to tell whether the next arrival will be before t or after t that I cannot decide without looking ahead, but at S N t + 1 I know that have exceeded t. So, N t + 1 is a stopping rule and is this finite with probability 1 for every t N t is finite so J is finite with probability 1 and expected N t is also finite.

So, expected J is finite so we can invoke Wald. Note that expected J = expected N t + 1 is finite for all t. By Wald we get what do we get? Expected S J = x bar times expected J.

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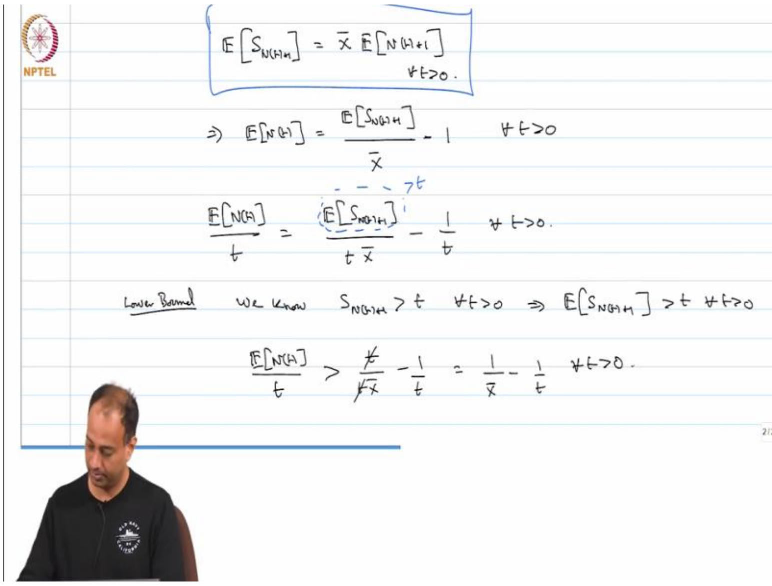


Now I can put, but I know what J is. So, I get expected S N t + 1 = x bar times expected J which is expected N t + 1. So, this equation is important this is true for all t greater than 0. So, what I have written I have just invoked Wald's equality for the stopping rule J = N t + 1. So, you can see that there is some expectation of N t is emerging here. So, I can rearrange this.

So, I want to deal with this expectation of N t I can write as can you help me out here expectation of S N t + 1 over x bar – this is true for all t greater than 0. What I want is expected N t over t as t becomes very large. So, expected N t over t = expected S N t + 1 over t x bar – 1 I think – 1 over t. I hope my algebra is correct. Now I really want to send t to infinity, but I do not know how to control this term expected S N t + 1.

So, now see one direction is very easy so if I want to prove that the limit of the left hand side is equal to 1 over x bar I basically want an upper bound and a lower bound on the left hand side. It turns out that the lower bound is actually quite easy. So, let us look at the lower bound. See we know that strictly greater than t for all t always the case. You can go back and look at the picture if you want S N t + 1 always occurs strictly after t.

Thus, expected S N t + 1 is greater than t. So, the right hand side so I have the following. So, I expected N t over t is equal to all this, but I can say is greater than or equal to t over t x bar – 1 over t. What have I done? See if you look at this term so if you just look at this term that I am saying is bigger than or equal to t actually it is bigger than t actually I can write bigger than here.

We can confidently write bigger not just bigger than or equal to. I am just putting a lower bound of course I have a cancellation here t is greater than 0 this is equal to.

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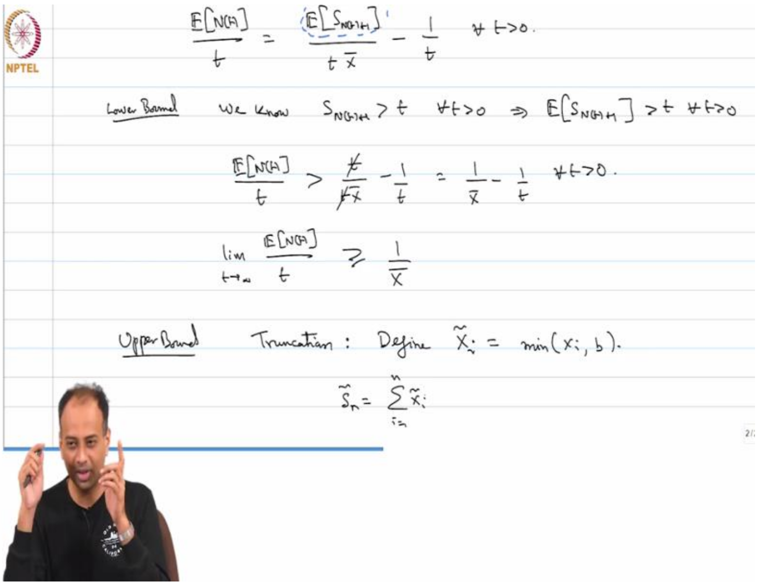

So, what do I have? I have expected N t over t is strictly greater than 1 over x bar – 1 over t. So, if I take limit expected N t over t should be greater than or equal to. See basically what I have is expected N of t is strictly greater than 1 over x bar – 1 over t for all t. If I send t to infinity greater than will become greater than or equal to. It is like this 1 over N is strictly greater than 0.

But if you send N to infinity the limit is greater than or equal to 0 you cannot say it strictly greater than 0. So, this I will get is greater than or equal to limit t tending to infinity of the right hand side which is this guy goes to 0 1 over t goes to 0. So, we will just get 1 over x bar. So, lower bound is done. So, whatever this limit is greater than or equal to 1 over x bar I want to prove equal to 1 over x bar.

I have proven greater than or equal to 1 over x bar. The upper bound is little more tricky because so I want an upper bound on expected S N + 1 – t that is what I got. So, I want an upper bound on this bit this is what is S N t + 1 – t. I want an upper bound on this expectation. Now, this is basically I want an upper bound on the expected value of the residual time what we have already talked about as the residual time.

But we know that the expected residual time depends on the second moment of x which could be infinite I have not made any assumptions about it. So, I may not be able to directly bound it. I have not said that second moment is finite or any such thing. So, in fact I should be able to prove elementary renewal theorem regardless of the second moment being finite or infinite. In fact even I am allowing x bar to be finite or infinite.

So, it is not clear directly how you bound this expected S N + 1 – t upper bound this guy. So, it is a little more involved, but we can use a standard trick in probability which is to something can be unbounded what do you do its standard trick in probability is to truncate the random variables. Some of the most important theorems in probability including the loss of large numbers and all that at some point you truncate to control things so a truncation trick.

Define x i tilde as min or max min of x i, b or maybe I should say max I am truncating it. So, I want it to be x i smaller than b and b when x i is bigger than b. So, I should put what I put towards right I always have this confusion min or max so this is good that is what I am defining and likewise I define S n tilde = sum over x i i = 1 to n of the tilde variable. So, you are like considering a new renewal process.

This tilde process where your x i tilde are equal to x i, but if x i becomes very large it becomes b x i becomes b and likewise the S n tilde is defined as the sum of first i and x i tilde and you can define a corresponding N tilde of t.

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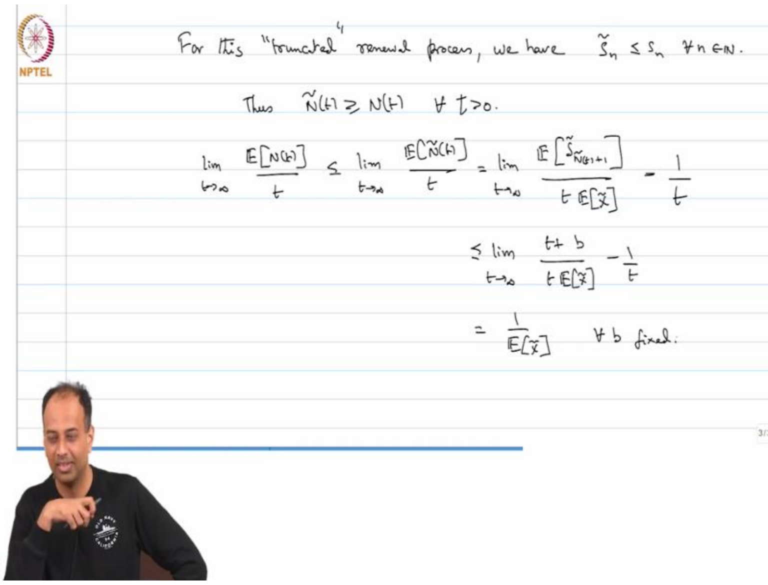


For this truncated renewal process we have S n tilde is what is the relationship between S n tilde and S n. See the x i tilde are smaller than or equal to the original x i's. So, S n tilde the nth arrival epoch in this tilde process should be lesser than or equal to S n. Now what can you say about N tilde t? So, in the tilde process the arrivals happens sooner or later they happen sooner which is why the epochs are smaller.

So, if the arrivals happens sooner than N tilde t must be they should be more arrival at a particular time because some of these random variables might have been truncated. So, in fact between using the equivalence between S n and N t you can prove that because if N t is greater than or equal to n then S n is less than or equal to t which means S n tilde is less than or equal to t which implies N tilde t is greater than or equal to n.

So, there should be more arrivals in this tilde process. So, I have some kind of an upper bound going now N t is upper bounded by some other thing so this is great. So, limit t tending to infinity expected N t over t is less than or equal to limit t tend to infinity expected N tilde t over t this has to hold. This relationship is true for every t so it has to hold in the limit. Now you do the same Wald trick.

So, you have this guy right expected N t over t is all this equation. So, you copy that down for the tilde process. This is equal to S tilde so this here I have to write expectation of x tilde. See I am using this equation out here for the tilde process so I will get – 1 over t. This is again true for all t greater than 0. Now, what is the saving graze about this tilde process? For the tilde process the S tilde N t + 1 N tilde t + 1 is at most t + b.

You should go back to the picture for tilde process. See the previous arrival occurred at t or before. So, the next arrival this gap cannot be larger than the largest possible inter arrival time in the tilde process which is b. So, here it is controlled the tilde process has been controlled it is truncated. So, this I can say is less than or equal to I can just keep this limit t + b over t expectation of x tilde.

And this guy of course goes away 1 over t goes away maybe I can just write it so I do not know why I am writing for all t greater than 0. I am taking limit t tending to infinity so there is no for all t greater than 0 I just rid of that. So, what is this equal to? T + b over t goes to what 1 correct. So, I have an upper bound of 1 over expectation of x tilde b is fixed this is true for all b fixed. So this much you will agree.

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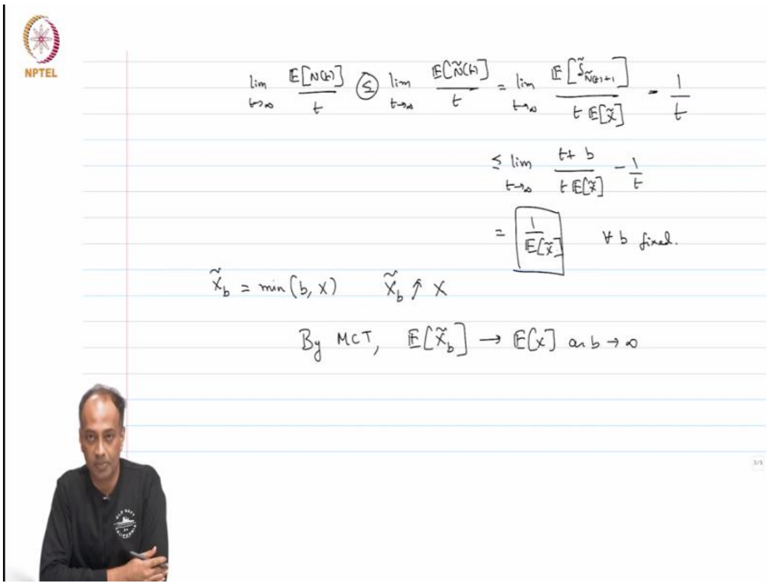


Now, it is useful to think of this x tilde as being indexed by b basically x tilde b is just minimum b, x and as b increases so we have x tilde b increases monotonically to x. So, by the monotone convergence theorem we can asset that expectation of x tilde b converges to expectation of x as b tends to infinity. So, in this inequality that expectation of N t over t is less than or equal to 1 over expectation of x tilde b for all b fixed.

We can send b tends to infinity and as b tends to infinity 1 over expectation of x tilde b will converge to 1 over expectation of x. So, 1 over expectation of x is also an upper bound on expectation of N t / t. We have always showed the lower bound so which proves the elementary renewal theorem.