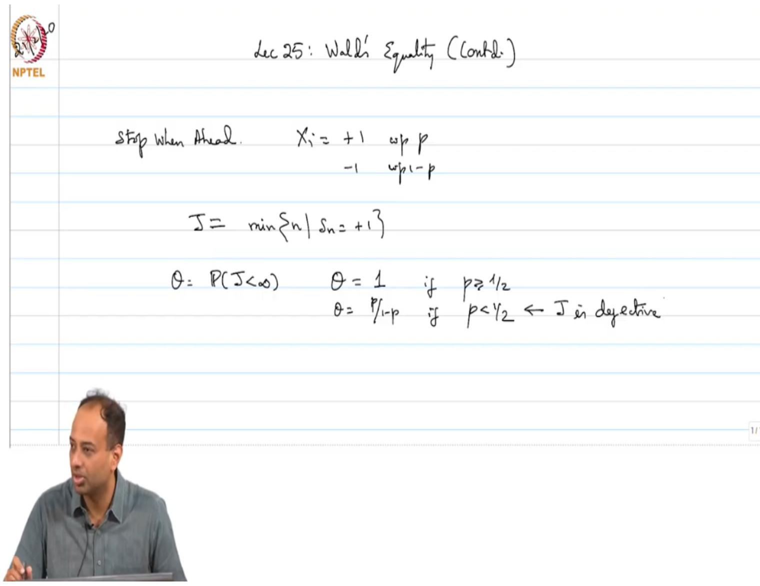
Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology - Madras

Module - 4
Lecture - 31
Wald's Equality (Contd.)

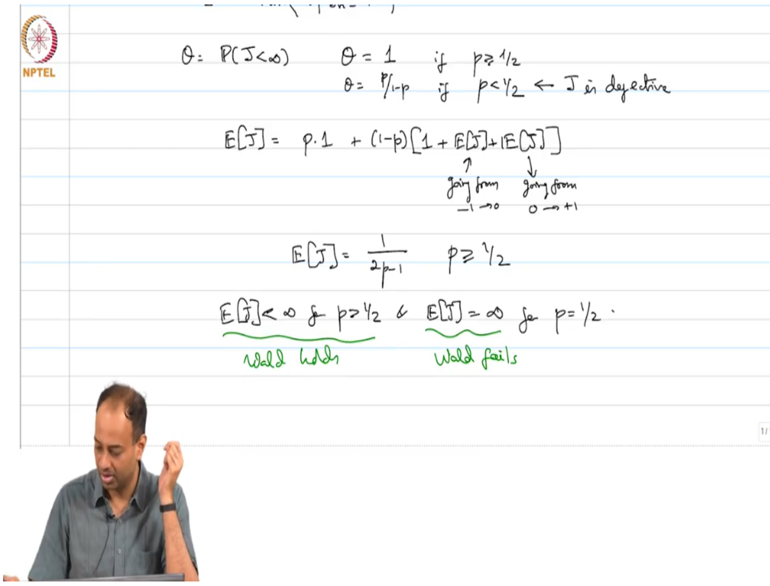
(Refer Slide Time: 00:15)



We were discussing Wald's equality and we were doing this example about stop when you are ahead. We have $X_i = +1$with probability p or $X_i = -1$with probability $1-p$ respectively. And you stop $J = min\{n: S_n = +1\}$. The first time you hit +1, you stop. And we said $\theta = P(J < \infty)$ is equal to probability that you stop eventually. And we said theta satisfies either; we wrote out a quadratic equation, right?

We said $\theta = 1$ if $p \geq \frac{1}{2}$; and $\theta = \frac{p}{1-p}$if $p < \frac{1}{2}$. This is where we were yesterday. So, if your probability of winning is at least half, then you are guaranteed to stop with probability 1. If your probability of winning is less than half, the probability that you eventually stop is only $\frac{p}{1-p} < 1$, which is less than 1, for $p < \frac{1}{2}$. Therefore, in this case, for $p < \frac{1}{2}$, J is a defective stopping rule, because it can be infinite with positive probability.

(Refer Slide Time: 01:55)

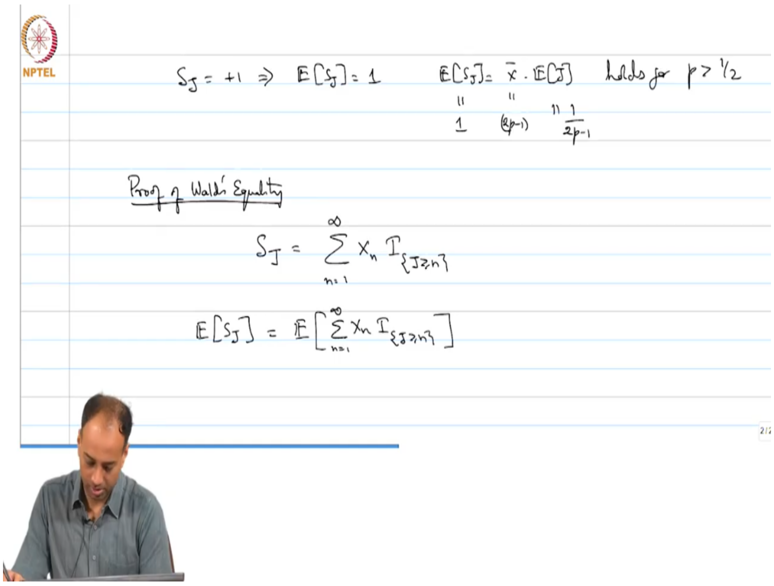


And similarly, you can calculate $E[J]$. You can write a recursive equation similar to θ as follows: You can write, $E[J]$ is equal to; so, J = 1 with probability p; because, if you get +1 straightaway, you stop at J = 1. So, it is $p \cdot 1$ plus; if you go down; so, you have gone down to -1, then what is the expected time to get from -1 to +1? It is the expected time to go from -1 to 0 plus the expected time to go from 0 to 1; but these are both equal to $E[J]$.

Similar sort of an argument as θ. So, you have $1 - p$; you go down once; it is already 1 step that you are taking; 1 + $E[J]$ + $E[J]$. So, this is for going from -1 to 0. This is for going from 0 to +1. So, from this, you can get $E[J]$ is equal to $\frac{1}{2p-1}$. This is for; if $p < \frac{1}{2}$, of course, $J = \infty$ with positive probability. Therefore, $E[J]$ will of course be ∞.

But if $p \geq \frac{1}{2}$, you get this expression. So, bottom line is that $E[J] < \infty$for $p > \frac{1}{2}$ and $E[J] = \infty$ for $p = \frac{1}{2}$. So, the case $p = \frac{1}{2}$ is particularly interesting. So, it is a symmetric up and down walk, +1; -1. So, here, you are guaranteed to eventually hit +1, with probability 1 you will stop; but the expected time for stopping is infinite. So, in this case, Wald fails. In this case, Wald holds, Wald equality holds. Why? Wald's equality demands that $E[J]$ be finite. In fact, you can check, right? What isS_J?

(Refer Slide Time: 04:57)



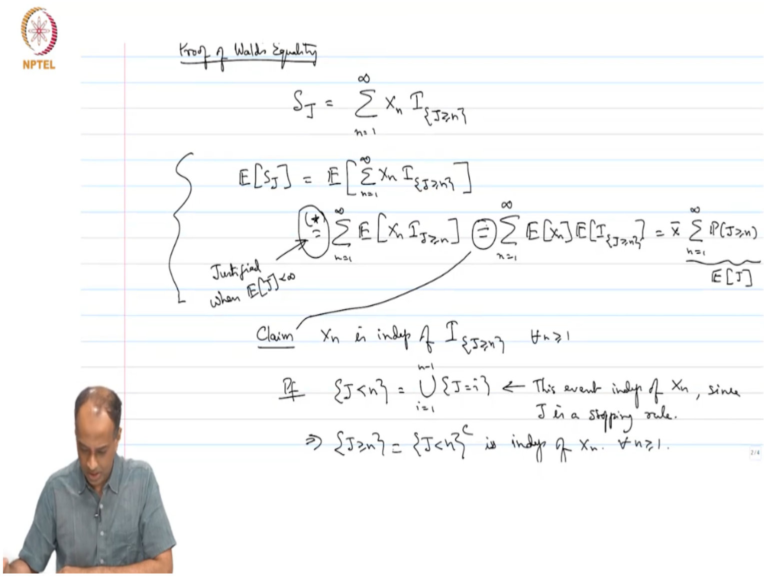
If you look at it, S_J is always +1. So, $E[S_J] = 1$. So, the question is, Wald says $E[S_J] = \overline{X} \cdot E[J]$. If this were infinite; see, if $p = \frac{1}{2}$, $E[J] = \infty$ would be infinite; $\overline{X} = 0$ wou; $E[S_J] = 1$. What is $\overline{X}$? $\overline{X} = 2p - 1$. So, $E[J] = \frac{1}{2p-1}$, as we calculated. So, it works. So, holds for $p > \frac{1}{2}$.

$p = \frac{1}{2}$, you will have a problem; you will have $1 = 0 \cdot \infty$, which is an absurd statement. So, $E[J] = \infty$; so, even if J is a legitimate stopping rule, if your $E[J]$ is infinite, you cannot apply Wald; $E[J]$ is finite, you can apply Wald. So, shall we go ahead and prove Wald's equality? The proof has a couple of subtleties. So, I will point out what happens.

There are a couple of points where you have to be very careful. You first write this; $S_J = \sum_{n=1}^{\infty} X_n I_{\{J \geq n\}}$, because; so, you are basically; J is the stopping time. So, you are summing all those X_i's which less than or equal to the stopping time, which is what S_J is. Now, you want to calculate$E[S_J]$, which is nothing but $E[S_J] = E[\sum_{n=1}^{\infty} X_n I_{\{J \geq n\}}]$; so, I have just taken what S_J is; I have taken expectation on both sides.

Now, comes the first subtle point where I want to take the expectation inside the sum; but the issue is that this is what kind of a sum? Infinite sum. So, you cannot always take a expectation inside an infinite sum. See, you can do it if the summands are non-negative, by monotone convergence theorem; but that may not be here; summands may not be non-negative, X_n 's could be negative, like in the previous example. So, you cannot just take it in.

(Refer Slide Time: 08:24)



So, I will eventually show that this works. I will write $\sum_{n=1}^{\infty} E[X_n I_{\{J \geq n\}}]$. This is justified. So, let me call this (*); justified when $E[J] < \infty$. I will justify this later. Just take it for now; we will get back to this and prove that this (*) is justified. So, next claim is that X_n is independent of $I_{\{J \geq n\}}$, $\forall n \geq 1$. Why is this true? Basically, it is the stopping rule condition.

You take the event $\{J < n\}$ which is simply $\{J < n\} = \cup_{i=1}^{n-1}\{J = i\}$. So, $\{J < n\}$ is an event that I stop before n, strictly before n, which is the event that I stop at 1 or stop or 2 or stop at $n - 1$. Now, the event, each of these $\{J = i\}$, the event that I stop at i, let us say, stop at $\{J = 2\}$ or $\{J = n - 1\}$ is independent of X_n. Why? Stopping rule. So, each of this is independent of X_n, since J is a stopping rule.

So, the event that I stop strictly before $n$can have only to do with X_1.through X_{n-1}.; it has to be independent of X_n. Now, so, this is the slightly mind-bending sort of a thing. So, you agree that the event $\{J < n\}$ is independent of X_n; I think that is intuitive. But what is the complement of the event $\{J < n\}$? $\{J \geq n\}$. So, if an event A is independent of; yeah, A compliment is also independent of X_n. This we know from basic probability.

So, this implies $\{J \geq n\}$ which is simply the complement of $\{J < n\}$, is independent of X_n. This is true for all $n \geq 1$. So, this makes sense, except that if you think about it, this is a little bit of a confusing thing. See, $\{J \geq n\}$is basically what? $\{J = n\}$ or $\{J = n + 1\}$ or $\{J = n + i\}$ in general. See, the event $\{J = n\}$ is certainly not independent of X_n. In fact, $\{J = n\}$ has a lot to do with X_n.

Usually, whether you stop at n or not will depend a lot on X_n. Similarly, the event $\{J = n + 1\}$ or $\{J = n + 2\}$ does depend on X_n. So, then, how is it that $\{J \geq n\}$ is independent of X_n? See, $\{J \geq n\}$ means $\{J = n\}$ or $\{J = n + 1\}$ or $\{J = n + 2\}$ and so on. So, none of these events are independent of X_n, but it turns out that $\{J \geq n\}$ is independent of X_n.

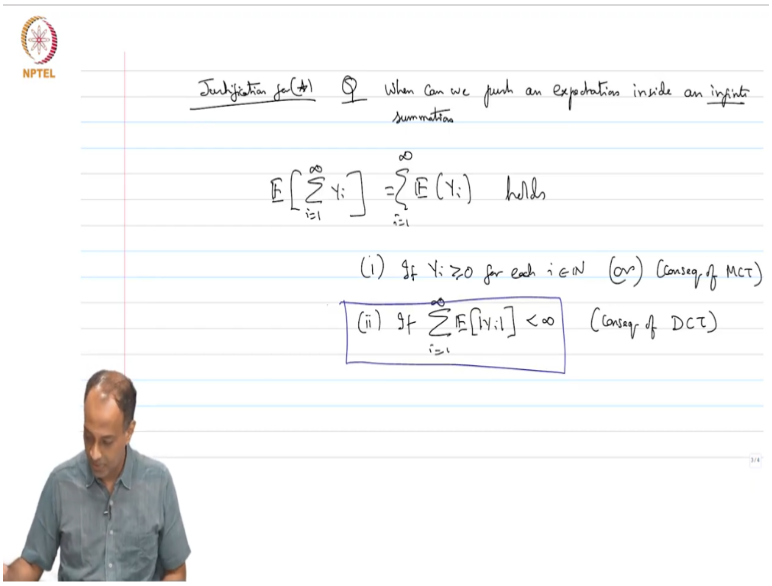
So, this is actually correct; there is nothing crazy going on here. The event $\{J = n\}$ or $\{J = n + 1\}$; so, where you stop, n or beyond, depends on X_n, but whether J is greater than or equal to n is independent of X_n. So, it is much easier when you go to $\{J < n\}$, and then take compliments; then it makes perfect sense; but if you try to argue directly, you seem to think, you cannot easily argue it.

So, but what I have written is clear, the event of stopping strictly before n is independent of X_n. So, $\{J < n\}$ is independent of X_n; therefore, $\{J \geq n\}$ is independent of X_n. So, that is great. So, then, what can I do here? So, the event $\{J \ge n\}$; so, the $I_{\{J \ge n\}}$ is an independent random variable from X_n. And, independent random variables are always uncorrelated.

So, $E[XY]$ can be written as $E[X] \cdot E[Y]$. So, this just becomes $\sum_{n=1}^{\infty} E[X_n]E[I_{\{J \ge n\}}]$; so, this is because of the claim. Now, what is the expectation of X_n? $E[X_n] = \overline{X}$; it is the common $\overline{X}$, that comes out. What is the $E[I_{\{J \ge n\}}]$?

Expectation of indicator is probability. So, that is $P(J \ge n)$. And what is this? This is the integral of the CCDF for this non-negative random variable. So, this just becomes integer value random variable, right? So, this is just $E[J]$. So, you are done, except that; so, what is the only thing that you have not proven here? The star, the justification for taking the limit inside. So, if these things do not cause you sleeplessness, you can just take it as true; but if it does make you sleepless, we will justify it. Ideally, it should make you sleepless.

(Refer Slide Time: 15:40)



So, we have proved Wald's equality, except for this justification for the step star above, which basically involves pushing an expectation inside an infinite summation. Now, pushing an expectation inside an infinite summation is not always justified, we need to impose some technical conditions under which this holds. And we have to basically prove that, under the assumptions of Wald's equality, the pushing in of the expectation inside the infinite sum is justified.

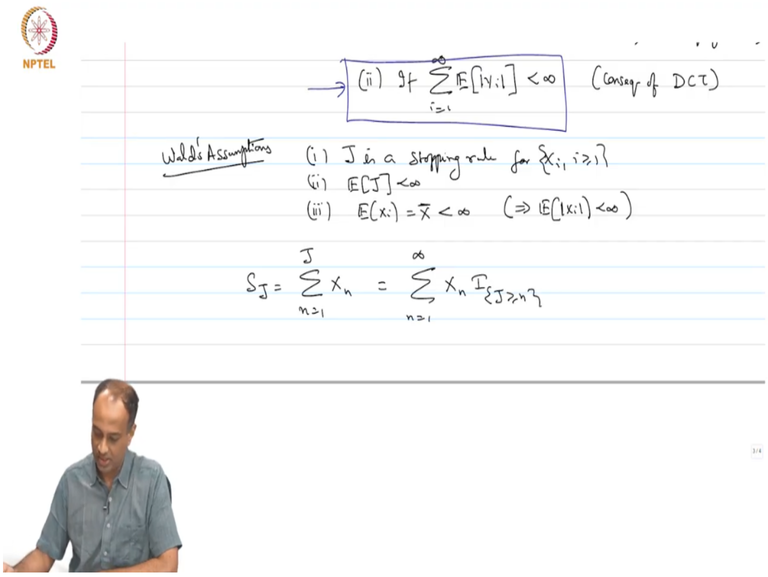
So, let us do that. So, the basic question which we are looking at is the following: When can we push an expectation inside an infinite summation? Of course, for a finite summation, you can always push the expectation in, because the expectation is linear, $E[X + Y]$ is always $E[X] + E[Y]$; and this is true even for a sum of n random variables; but you cannot always push an expectation inside an infinite sum.

So, essentially, the question we are asking is this: When is it justified to say that; let us say, Y_i's are some random variables; when can I say that the $E[\sum_{i=q}^{\infty} Y_i] = \sum_{i=1}^{\infty} E[Y_i]$? When is this justified? So, this holds; this is not always true, but it holds under 2 prominent sufficient conditions. The (i) condition is that, if $Y_i \geq 0$ for each $i \in N$. then you can push the expectation inside the infinite sum.

The other sufficient condition is, if $\sum_{i=1}^{\infty} E[|Y_i|] < \infty$, then we can still say that you can push the expectation inside the infinite sum. So, this holds under this or this; you do not need both. If either 1 holds or 2 holds, then you can push the expectation inside the infinite sum. The condition 1 is a consequence of the Monotone Convergence Theorem, and the sufficient condition 2 is a consequence of the Dominated Convergence Theorem.

You can try and prove this as an exercise. I have given you the hint on what major result you have to use. So, we are going to use this sufficient condition to justify the pushing in of the expectation in the (*) here. So, let us justify this equation (*).

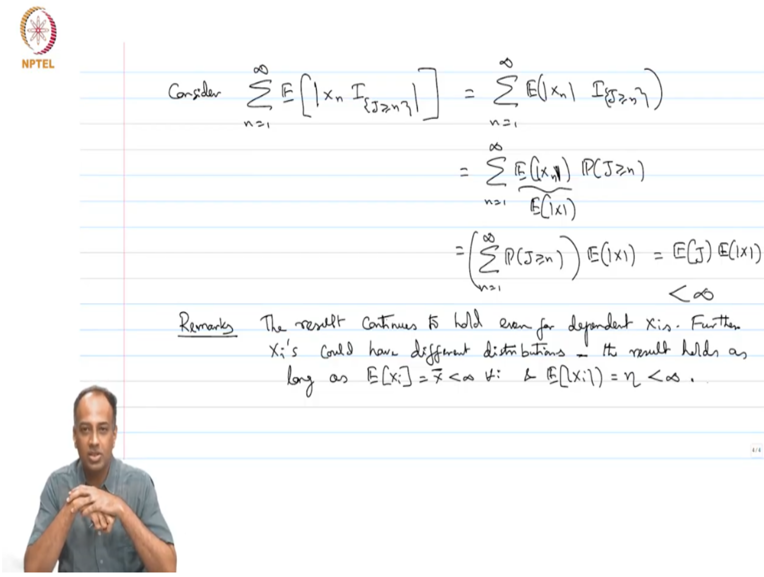
(Refer Slide Time: 19:19)



Before that, so, let me remind you of the Wald's assumptions. So, the Wald's assumptions are that J is a stopping rule for $\{X_i, i \geq 1\}$. We have assumed also that $E[J] < \infty$. And we have assumed that $E[X_i]$ is some common mean $\overline{X} < \infty$. You need a finite mean for these X_i's. This basically implies that $E[|X_i|] < \infty$.

So, we will use these to prove that condition 2 as written out here, holds for the sum S_J. So, recall that S_J whose expectation we want is just $S_J = \sum_{n=1}^{J} X_n$, which can be written as $S_J = \sum_{n=1}^{\infty} X_n I_{\{J \geq n\}}$. This we have already seen. So, the question is, can I take the expectation inside this infinite summation?

(Refer Slide Time: 21:10)



And one sufficient condition is 2, which is to look at, consider this sum, consider $\sum_{n=1}^{\infty} E[|X_n I_{\{J \geq n\}}|]$. This is my expectation of absolute Y_i. Of course, the absolute value of an indicator is just the indicator itself; so, this is just $\sum_{n=1}^{\infty} E[|X_n| I_{\{J \geq n\}}]$.

And by the stopping rule property, I already know that absolute $|X_n|$ is independent of $I_{\{J \geq n\}}$. We proved earlier that X_n is independent of $\{J \geq n\}$; by the same argument, you can prove that $|X_n|$ is independent of $I_{\{J \geq n\}}$. So, this is equal to $\sum_{n=1}^{\infty} E[|X_n|] P(J \geq n)$

So, this will become $E[|X_n|] \cdot E[I_{\{J \geq n\}}]$, which is just $P(J \geq n)$. So, now, this $E[|X_n|]$ is assumed to be finite, right? I am going to assume that this is some finite common number. Let us say this is$E[|X|]$, which is some finite thing. So, this becomes $(\sum_{n=1}^{\infty} P(J \geq n))E[|X|]$,

And this is nothing but $E[J] \cdot E[|X|] < \infty$. Now, we have assumed that both $E[|X|]$ and $E[J]$ are finite, so that this product is finite. So, this implies that 2 is satisfied, the condition 2 which I said; if you ensure this condition 2, then you can push the expectation into the infinite sum.

So, since we have proved that for S_J, you can push the summation in. Thus, as we did in (*); that is justified. So, that basically completes the justification. And therefore, the proof of Wald's equality is complete. Now, I want to make a few remarks. See, we have assumed the statement of Wald's equality; we have assumed the X_i's to be IID random variables, and J to be a stopping rule for these random variables; but nowhere in this proof is the independence actually used.

So, X_i's could be dependent, that is the first thing to note. So, the proof and the result continue to hold as though, even for dependent random variables. Further, we have also assumed identically distributed random variables, which is again not needed. What we need is a common mean $\overline{X}$which is finite. So, X_i's could even have different distributions; but as long as for each of these X_i's have a common $E[X]$ and $E[|X|]$, both be finite values, then this continues to hold.

So, maybe I should say so. The result continues to hold even for dependent X_i's. Further X_i's could have different distributions. The result holds as long as $E[X_i]$is some common value $\overline{X} < \infty$, $\forall i$, and $E[|X_i|] = \eta < \infty$. So, nowhere have we used independence, and even identically distributed is not needed.

And if you look at this proof even more closely, the key step was that bit, that equality, where we used the stopping rule property, where we showed that the event $\{J \geq n\}$ is independent of X_n due to the stopping rule property. Actually, we have just used the uncorrelatedness of X_n with $I_{\{J \geq n\}}$. Of course, uncorrelatedness follows from independence, which follows from stopping rule property; but really what you need is uncorrelatedness for each n between X_n and $I_{\{J \geq n\}}$.

So, that is as general as you can get. In fact, there are statements of Wald's equality even when the X_i's do not have a common mean $\overline{X}$. You can make a more general statement which is a little bit more messy. In fact, Wikipedia has a more general statement of Wald's equality. But, we can just stop with this particular version of Wald's equality, with just the understanding that although we have stated it for IID random variables, you do not need independence, you do not even need identical distribution, but you do need $E[J]$ to be finite and $E[X_i]$'s should be some common value $\overline{X}$ which is finite. That is all that is needed. So, that concludes the proof of Wald's equality.