Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology - Madras

Module - 4
Lecture - 29
Stopping Time

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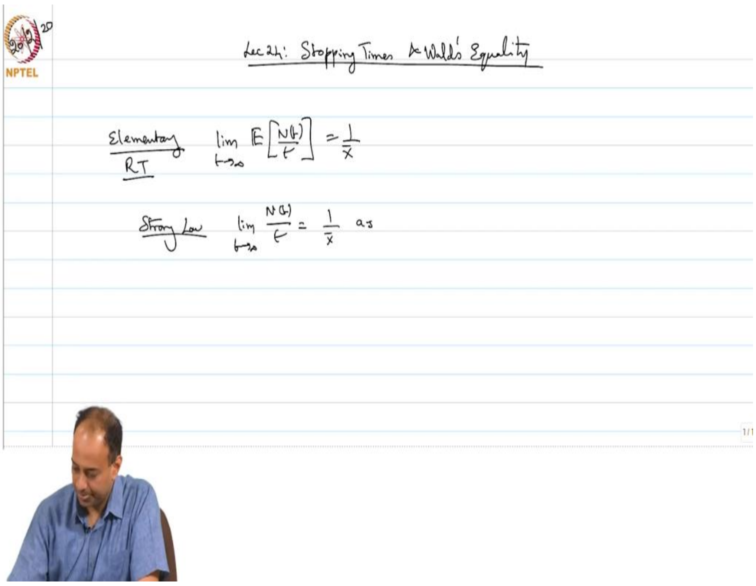

Welcome back, good morning. We are heading towards the elementary renewal theorem, which is the following statement. For a renewal process, elementary renewal theorem says limit t tending to infinity expectation of N t over t is equal to 1 over X bar. This is not a direct consequence of the strong law, as I already mentioned. And the proof of this is somewhat non-trivial; so, we need to build up some machinery to prove this.

So, somebody pointed out to me yesterday; I think I should tell you this. So, you have; see, strong law says this; says limit t tending to infinity N t over t is 1 over X bar, almost surely. So, if you look at this N t over t as a sequence of random variables indexed by t; it is an uncountable index; but you have this almost sure limit of 1 over X bar; so, elementary renewal theorem, it just looks like you are looking at the expectation of the sequence.

Normally, it is not the case that if you have a sequence X n converging to some X almost surely, it is not at all the case that expected X n converges to expected X; not true. There are conditions under which this is true. The most famous celebrated condition under which you can, so to speak, interchange the expectation in limit is the monotone convergence theorem. So, if X n converges to X almost surely and monotonically, then expectation of X n converges to expectation of X.

And then, likewise, there is a dominated convergence theorem, if X n is dominated by a random variable whose expected value is finite, then you can again interchange. Neither is true for N t over t. N t over t as a sequence in t; I drew some sample paths, right? So, it jumps and then goes down and then jumps and then goes down; it is not monotonic. It is certainly not monotonic in t and neither is it dominated in t.

So, if either of this were true, we can just apply one of these hammers MCT or DCT and get elementary renewal theorem; but it is not so simple. It is true, but it requires a completely different proof. You cannot just say strong law followed by MCT or DCT; does not work. So, I just wanted to make that remark. Today's lecture is not directly about elementary renewal theorem. We are heading there, but we want to build up some missionary; in particular, we want to talk about stopping times or stopping rules; or stopping rule is the same.

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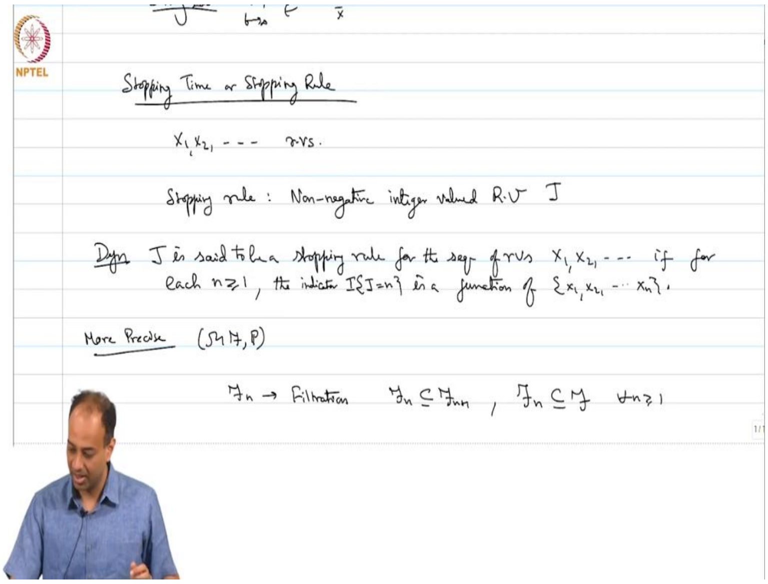

So, I want to discuss this topic of stopping times. So, this discussion on stopping rule or stopping time has nothing to do with renewal processes. It just has to do with probability and random variables, it does not have to do with renewal processes. We will use stopping times in a particular way on a renewal process in trying to prove the elementary renewal theorem; but before that, I have to tell you what a stopping time is; that is all.

So, intuitively, stopping time is a, first of all, it is a non-negative integer valued random variable. So, it is the time; when we are looking at, let us say discrete time; let us say you have some random variables which are realising X 1, X 2, ..., which are realising in discrete time. So, these X 1, X 2, ... could be some sequence of your winnings in some, let us say, some casino.

There is some X i's; you play some game in a casino each time, you win an amount X i. This X i could be; if it is positive, you won something; if it is negative, you have lost something. Now, you want to decide when you are going to stop playing. So, typically, if you go to the casino with, let us say a 100 rupees, you want to say that I will quit playing either when I win a certain amount or I go totally bust; of course, if you go bust, you have to stop; or you may just be happy by making another 50 rupees or whatever; you may just decide when to stop.

The key issue is that, the time when you decide to stop is a function of the rewards that you obtain, the X i's that show up. Let us say I stop at some time n. The time n that I stop is itself a random variable, and is a function of what these previous X i's were. You see that? So, this n is not some independent random variable. The random variable index of time that you stop playing is certainly dependent on the values of the X i's you have seen.

So, if you had a bunch of very good winnings, you may just make 50 rupees more and say, I am very happy, I will stop; or you may just lose all your money and just get out. So, when you stop? The index of time that you stop is very much a function of the; is data dependent, so to speak; it is dependent on the X i's that you see. This kind of a situation is called as stopping time. This is all there is; intuitively, this is all there is.

What is the key issue in a stopping time? The key issue is that, whether you decide to stop at a particular time or not to stop is only a function of the random variables you have seen so far. Suppose I told you ahead of time that; you played for 21 times; I tell you, the twenty-second time, you will get a very good reward. You will tend to not stop, right? You will play. Or if I tell you that twenty-second time you are going to get ruined; you will stop.

But this is not a stopping rule. Why? Imagine you could do this, right? Imagine you had some extra information about your slot machine or whatever it is that you are playing. If you knew what is going to come, you can really make a lot of money out of this; and of course, casinos will hate you for this, right? So, but you can never really do this. If you had some side information on what the slot machine is going to produce, you can look ahead and decide to stop or not stop.

This is not a stopping rule, because you are looking ahead; that is not allowed. So, a stopping rule is a, this non-negative integer value random variable, it is the time that you stop; that is why it is called a stopping time. You stop based on what you have seen so far. It is only a function of what you have seen so far. You are not allowed to look into the future. This is all there is intuitively.

It is actually, intuitively very easy to understand what a stopping rule is, but one has to define it mathematically. So, that is what you will do. So, in this particular section on stopping rule; stopping rule and stopping time are the same thing; people use it interchangeably. And in this section, you can just forget all about renewal processes; it is just about some random variables.

So, I will give you a slightly informal definition. I will also give you a formal definition which involves some sigma-algebra and all that. If you know it, great; if you do not know it, it is okay; we do not need it. So, let us say that X 1, X 2, ... are random variables. So, stopping rule, we will denote by J. So, a stopping rule is a non-negative integer valued random variable denoted by J.

Definition: J is said to be a stopping rule for the sequence of random variables X 1, X 2, ... if for each n, the event J equals n is a function of X 1, X 2, ... till X n. Actually, some people put indicator J = n; maybe you can put an indicator here. It does not matter, right? Indicator of an event is 1 if the event is true, 0 otherwise; if the indicator is a function of X 1, X 2, X n. So, the key issue is that the event J = n is independent of the future X's. That is the key issue.

So, I have made this I, indicator J = n a function of X 1 through X n. This is enough for our purposes, but a more formal definition is given in terms of measurability of the event J = n by the sigma-algebras generated so far, by X 1 through X n. So, more precise definition is as follows. So, do you guys know what a filtration is? You guys do not know what a filtration is, right? So, let me just tell you; this is not really in your syllabus, but I will just tell you for the sake of completeness, okay?

So, you are given this; everything comes from Omega F P; Omega F P is the probability space on which all the actions take place. So, F n is called a filtration if; so, a filtration is basically a sequence of increasing sigma-algebras such that all these F n's are sub-sigma-algebras of F and these F n's get progressively bigger. So, you can think of; what is really happening here is that you have, let us say n is just time; as more and more random variables get revealed in time, your knowledge so to speak of the sample space on what event occur and what does not occur gets bigger and bigger; you get more and more information.

And this more and more information is captured by this sequence of increasing sigma-algebras, this F n's. And of course, all these F n's are sub-sigma-algebras of F; this F is a very big sigma-algebra which measures everything. And F n is basically the information available to you till n. So, a formal definition of a stopping rule:

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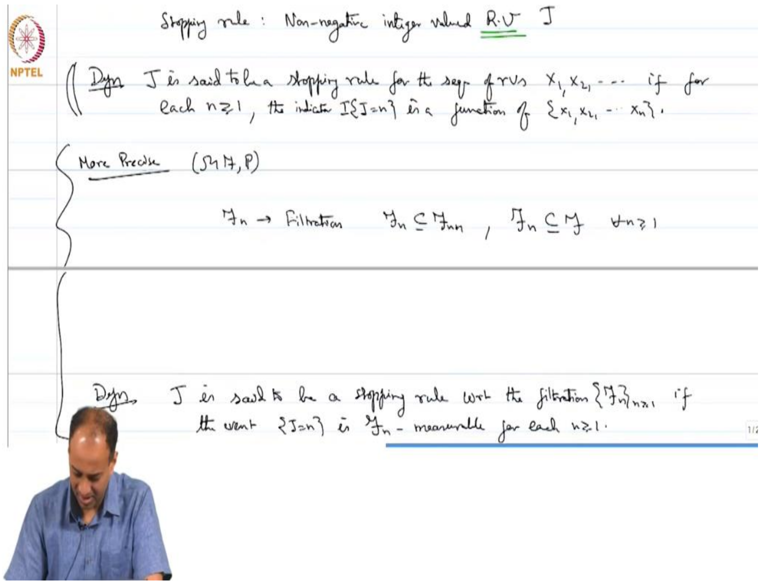

A more precise definition is that J is said to be a stopping rule with respect to the filtration F n if the event J equals n is F n measurable for each n. So, this is the precise definition of a stopping rule. It is a non-negative integer valued random variable, J. And for every n, the event that you stop at n, J = n, must be F n measurable. Remember that F n is the sum total of all information you have a time n.

So, using only the information available at time n, you have to decide whether you stop or not. So, that is what a stopping rule is. So, in our setting; so, this bit, going till here, this bit you can; I will not hold you responsible for this, knowing this measure theoretic definition, but it is good to know. If you are at all familiar with how these sigma-algebras capture information and these growing sigma-algebras can be used to capture information that is evolving in time, this is useful to know.

Now, for our purpose, this F n is simply the sigma-algebra generated by the random variables X 1 through X n. Given any set of random variables, I can talk about the sigma-algebra generated by those random variables. So, in our setting, the definition we will use, this guy, there, I am saying, indicator J = 1 is a function of X 1 through X n. What I really should be saying is, the event J = n is measurable under the sigma-algebra generated by X 1 through X n.

F n in that case is just sigma-algebra generated by X 1 through X n. So, this is the definition we will use practically. And everything that is below is just for you to know, and I will not, I do not assume; so, this course, we do not do too much information theory and all that. Good. So, now, there is one little remark I want to make. So, we have said that this stopping rule is a non-negative integer valued random variable.

So, by random variable, I implicitly mean that the random variable takes finite values with probability 1. The random variable is a map from omega to R; so, it has to take finite values with probability 1, which means that a stopping rule, you should be able to eventually stop with probability 1; only then, it is a stopping rule. See, a little more generally, you can talk about defective stopping rules which satisfies this property of not looking ahead, but it may not stop with probability 1; there may be a positive probability that you never stop.

So, for example, if you go to a casino with some amount of money; and typically, if you go to a casino, you are more likely to lose than to win, which is why casinos even exist. You will go and say that I will stop; you can borrow any amount of money; I will stop only if I double my money. You can show that in this kind of a setting, there is a positive probability that you may never double your money; you may just keep going down, because it is more likely that you will lose.

This kind of a setting, of course, you are not allowed to look ahead, but the issue is that, with positive probability, you may never double your money. So, that kind of a stopping rule is called a defective stopping rule; because, it is a stopping rule, yes; but it is a defective random variable, meaning that it can be infinite with probability greater than 0. So, such a stopping rule is usually called a defective stopping rule, because it is not quite a random variable, it can be infinite, plus infinity with probability greater than 0.

So, when I say stopping rule, it usually means it is a random variable, which means you stop with probability 1. And usually, there is a chance that you may not stop. If it is not clear a priori whether you are going to stop with probability or not, you can say possibly defective stopping rule. So, possibly defective stopping rule still satisfies this J = n is only a function of X 1 through X n, except that you may not stop with probability 1.

That is just a remark I wanted to make. So, if I give you an example; so, let us say you go with 100 rupees to a casino; you go saying that, I will stop after playing 5 times. Is that a stopping rule? This is a very trivial example. I am going to always stop; I am only going to play 5 times. Think about it. So, in this case, what? The random variable J is constant, is equal to 5.

Does this depend on future values? It does not depend on future values. I am not looking ahead. So, it should be a stopping rule, no? It is finite, obviously, it is finite with probability 1. J = 5, is that a stopping rule? No? See, in this case, J = 5, it is a constant random variable. A constant random variable is? Well, it is measurable under; if you look at this sigma-algebra language, it is measurable under any trivial sigma-algebra.

So, you can apply this more involved definition. Or if you just look at this, it is only a function of X 1 through X 5. It is a constant function of X 1 through X 5. I always stop at 5. So, I do not look ahead; that is all that matters. So, it is a stopping rule if I stop at 5. So, that is one trivial example. I will give you another example. I generate a, some non-negative integer valued random variable n ahead of time, independent of these X i's.

I do not even know what X i's are going to come, but I am going to generate n, let us say a Poisson random variable. And once I have that realisation, I will play so many times. So, where n; so, here n, let us say, in this case J, is some Poisson random variable or whatever non-negative valued random variable independent of the X i’s. Is this a stopping rule? See, I have generated some n; I am going to say that, I am going to stop at whatever n, whatever that random variable is.

Actually, I am not looking at any of the X i's, forget looking ahead. So, it is a stopping rule. **"Professor - student conversation starts"** Yeah, that is all there is. It is only a function of; So, when I say something like this, what I really mean is that, it should not be a function of future; it can be a function of your data so far; so far being the key issue. Then I could say for example that I go in with 100 rupees, I win 1 rupee with some probability P; I lose 1 rupee with probability 1 - P; I stop when I either make 50 rupees or go bust.

This is a stopping rule. You can show that this is a stopping rule. But you cannot, for example, what is not a stopping rule is to say that, I will stop if my next reward is something bad. That is not a stopping rule. I will stop at a particular time if my next reward is something bad, or whatever. In my example, if my next 2 rewards are minus ones, I will stop; not a stopping rule, because you have to look ahead. That is all there is. It takes some getting used to this concept. **"Professor - student conversation ends"**