Stochastic Modeling and the Theory of Queues
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Module - 4
Lecture - 28
Renewal Reward Theorem (Time Average) - Part 2

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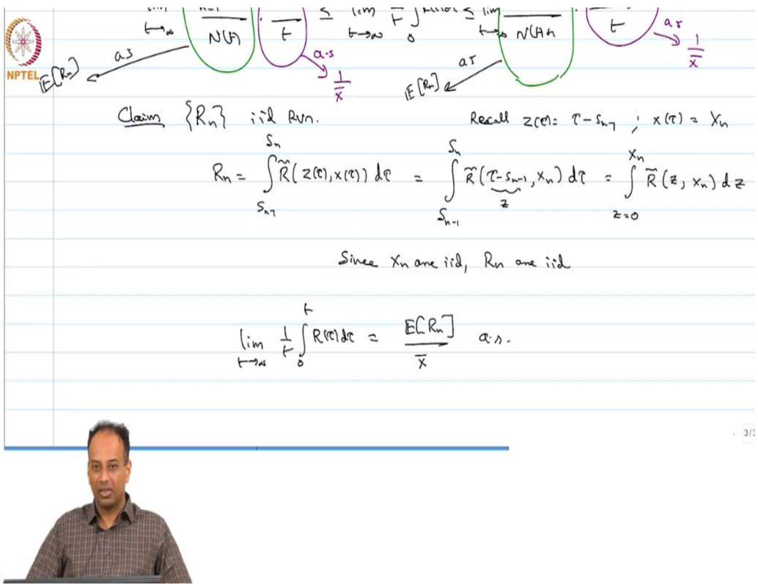

One thing we can do now, for example; we have calculated the average residual life and average age and so on; now, we can actually calculate the distribution of the age, or distribution of residual life, whatever you want. Now, how would you do that? So, you want to look at, let us say, the distribution of the residual life. You want to look at Y of t less than or equal to some little y.

That will be the distribution of; now, you are looking at the fraction of time, that this residual time is less than or equal to little y. What is the time average of such a reward? is what you want. So, therefore, you can take a reward function which is equal to 1 when Y of t is less than or equal to y, and 0 when it is not the case. And you find the time average of this. You can use the renewal reward theorem to get this time average. So, this is a good example.

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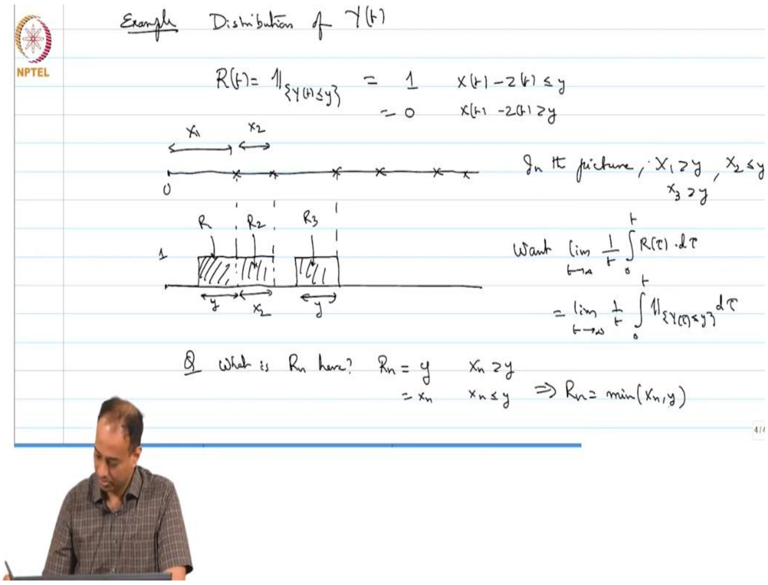

So, you want to take, let us say, R of t is just indicator. Now, if I take the time average of this function; see, little y is something fixed. I want the CDF so to speak of Y t. Little y is fixed. So, I am looking at the average fraction of time that Y t is less than or equal to little y, if I find the time average of this reward function. If you want to write R of t in the form R tilde Z t Yt, so, this is equal to 1 when X t - Z t less than or equal to little y, and 0 when X t - Z t is greater than y.

I am writing it as R of Z t, X t; that is what we have chosen. So, just to give you a picture, let us say, the 0, that is my renewal process, some particular realisation of renewal process, and little y is something fixed; little y you fix whatever you want. Now, my reward function R of t will be 1 whenever the residual life, the time to the next renewal is less than or equal to little y. So, at each time t; so, my time is, let us say here.

I will say, is my time to the next renewal less than or equal to y? If it is not, I will give 0 reward. At some point, let us say here, it will start becoming 1. And the width of this will be how much? Little y. So, when I am here, when my time is here, my residual life becomes less than or equal to y. y is some; you fix a little y, right? Then your reward will become 1; otherwise, it will be 0.

Again, it will become 0 here if it were the case that the next renewal interval is greater than y. Suppose, so, this is X 1. So, the way I have drawn it, X 1 is bigger than little y. That is why it was 0 here. And when the residual time became less than or equal to y, I got a reward of 1.

And the way I have drawn it, this X 2, I have drawn it to be smaller, deliberately. So, if X 2 were less than y, I will start accumulating a reward of 1 straight away.

And then, this renewal interval is much longer, so, my reward goes back to 0. Again it will become 1; so, this will be y. So, if you look at; now, what is this equal to? This is simply what? X 2. The way I have drawn it, X 2 is less then y. So, in the picture, X 1 is greater than y; X 2 is less than y; and X 3 is greater than y and so on. **"Professor - student conversation starts"** Because your reward becomes 0; see, at this time, what is your residual time? X 3.

But X 3 is bigger than; the way I have drawn it, X 3 is bigger than y; so, it becomes 0 immediately. And then it keeps at 0 until my residual time becomes less than or equal to y. Then it shoots up to 1 again. And the reason it did not fall down here is because I have drawn X 2 to be smaller than y. If X 2 were bigger than y, again it will fall down and pick up. **"Professor - student conversation ends"**

So, now, I want the time average reward of; so, want to calculate what? Limit time average fraction of the time that residual time is less than or equal to y. This is what I want. Integral 0 to t indicator of Y t less than or equal to y; I think I should write d tau; Y tau d tau. And this is the reward function in question. Now, what is R n now? Now, question: What is R n here? It will be the area under; so, this area is R 1; this area is R 2.

This is the analogue of the isosceles triangle areas, except the function now is different. And this guy will be R 3. So, R 1 is how much? Little y. R 2 is X 2. R 3 is y. So, what is R n? So, if your X n is larger than y, your R n is y. But if your X n is less than y, R n is X n. So, R n should be what? So, it should be y when X n is bigger than y, and equal to X n when X n is less than or equal to y. So, what is that R n? R n should be min or max? min. min of X n, little y. Finished.

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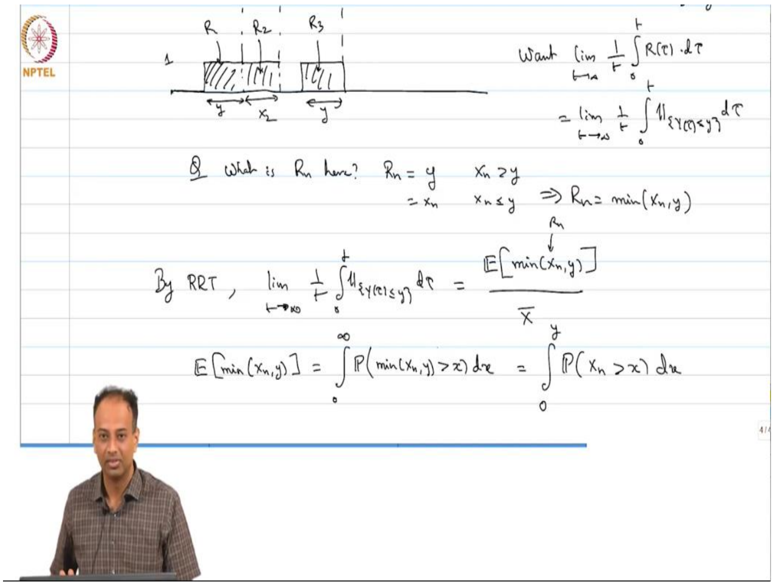

So, thus, by RRT, renewal reward theorem, I have the answer already. Limit t tending to infinity 1 over t integral 0 to t indicator Y tau less than or equal to little y d tau is almost surely equal to expected min X n y over X bar. Because R n is min X n, y. So, expected R n is; so, this guy is just R n; I have taken expected R n, that is all. Now, what is expected min X n y?

Now, expected min of X n, y is simply integral 0 to infinity probability that; y greater than x dx. Why? Non-negative random variable. Expected value is the integral of the complimentary CDF. So, this is clear. Now, what does this boil down to? So, this is just; so, this will become like this, right? This will become integral 0 to y probability that; or X has to be less than y. X cannot go from 0 to infinity, because you are looking at the event that min of X, y is greater than x; so, x cannot be; when x bigger than y, the integral becomes 0.

So, that is all that remains. So, when you have minimum of X, y, where if you are looking at values of x which are less than y, then min of X n, y will simply be X n. That is that. So, if you want me to draw a picture;

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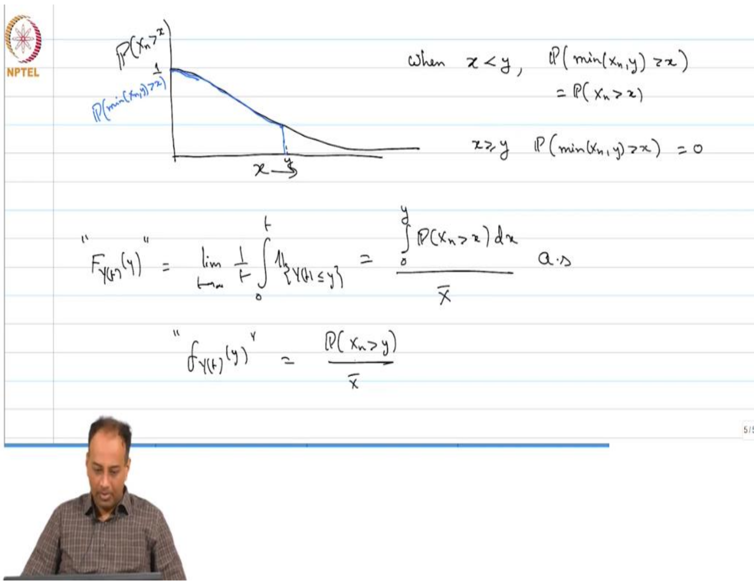

You are looking at this random variable, right? So, let us say this is my original CDF. It starts off at 1, then decays down to 0. It does not have to be a nice continuous function like this, but I have drawn something. So, this is the CCDF of X n. I want the CCDF of min of X n, y. So, if you have, let us say y is here. So, when x is less than y, so, what is the probability that min of X n, y is greater than x? It will simply be probability that X n > x.

And when x is greater than or equal to y, what is the probability that min X n, y is greater than x? 0. So, the CCDF; let me draw this in a different colour. So, the CCDF of probability of min of X n, y greater than x; if I draw that guy, it just follow this curve and then drop down to 0. That is what I have integrated. I am just justifying this step in this picture. So, perfect. So, let us go back to this.

So, we have limit t tend to infinity 1 over t. The time average fraction that my residual time is less than or equal to y is simply integral 0 to y probability X n > x dx over X bar. So, this you can think of as some sort of a CDF of Y t. But it is not really a CDF, because it is a time average fraction, which is why I put it inside quotes. At some level, it is the time average fraction that Y t is less than or equal to y.

Since time averages are generally equal to ensemble averages, you can look at it as the expectation of the indicator Y t less than or equal to y. And what is the expectation of an indicator? It is the probability of that event, probability Y t less than or equal to y. So, it is a little bit like saying, if the renewal process is running for a long time, what is the CDF, what is the probability that your residual life, residual time is less than or equal to little y?

It is this answer, integral 0 to y CCDF over X bar, almost surely. So, there is something quite nice about this. There are 2 remarks that I would like to make about this answer that we have obtained. See, the one is that, even if X n, this inter-renewal times are not continuous random variables; they could be discrete or mixed or whatever; the CDF of the residual life is the integral of some nice function.

So, we can actually speak of some sort of a density, with the same understanding that it is not quite entirely rigorous. It has a density of that form. So, the density of the residual age is like the CCDF of your X n's. See, the X n's do not have to have a density, but the residual time always has a density, if you believe that this time averages and ensemble average will be the same; because, I am just differentiating what I have above.

So, it is nice that even if you have a renewal process in which X n's are discrete or mixed or whatever crazy thing, the residual life will always be a sort of a continuous random variable, asymptotically, when t becomes large. That is one thing. The other thing is that, if you find the expectation of Y t now; the average; so, this is some distribution; what you get now? Think about it. First of all, is this a valid density? It has to integrate to 1.

If you integrate it from 0 to infinity, what happens? The integral of the numerator is? X bar; denominator is also X bar; so, you get 1. So, if you want to calculate the expected value of this distribution, what will you do? You will integrate y times P X n > y 0 to infinity. And that will be like your second moment; you can calculate using integration by parts. You will recover the answer for expected Y t or average of Y t, which is expected X square by 2 X bar; you will recover that from here. So, this is very nice. So, we have the distribution of residual time.

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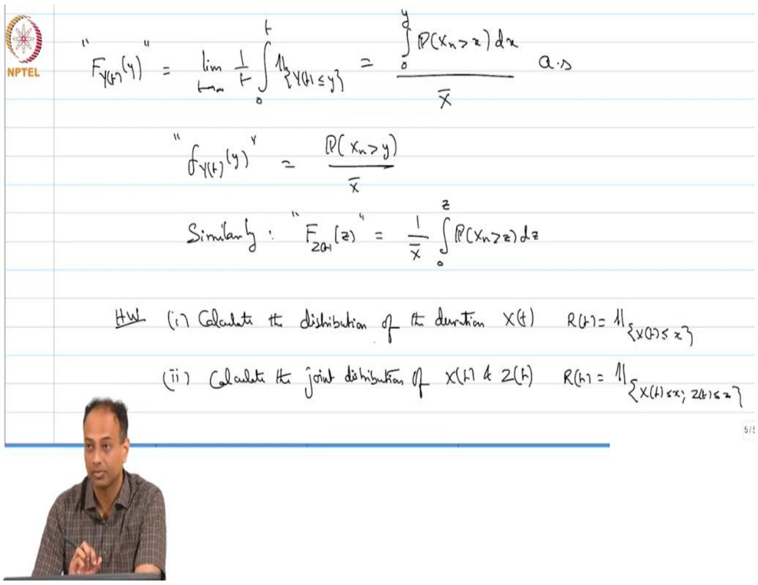

Likewise, you can actually also calculate; a similar calculation shows for the age; a same sort of an answer you will get; same argument. Homework: You can calculate the distribution of the duration X t. So, here you will take R of t equal to indicator X t less than or equal to some little x; same procedure. Second homework: Calculate the joint distribution of X t and Z t, or X t and Y t for that matter; does not matter.

For here, what will you take R of t equal to indicator X t less than or equal to X and Z t less than or equal to Z. This is the reward. You can calculate the entire joint distribution of these X's and Z or Y and Z. So, once you get the joint distribution, you should be able to do sanity checks like, can you recover the marginals? Please try this as a homework. It is an interesting exercise in renewal reward. So, I have done for today.