Stochastic Modeling and the Theory of Queues
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Module - 4
Lecture - 27
Renewal Reward Theorem (Time Average) - Part 1

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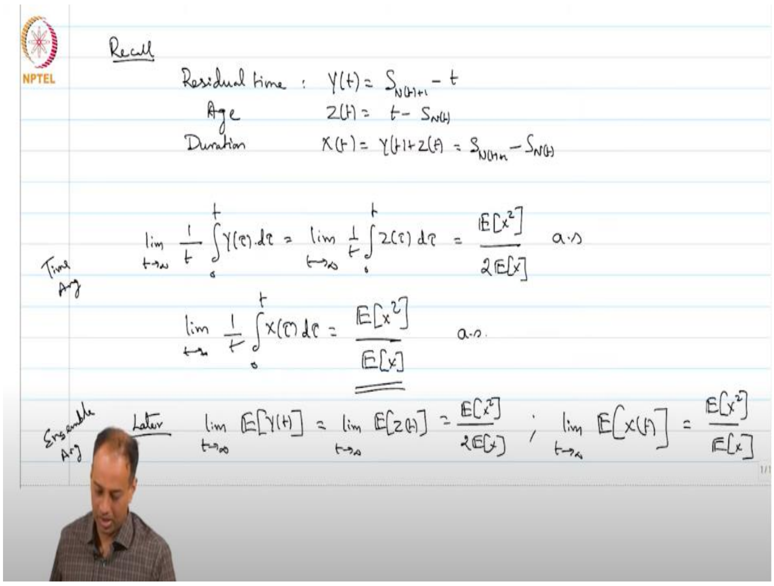

Welcome back, good morning. Recall from yesterday that we defined residual time as Y t as S N t + 1 - t; age as t - S N t; and duration, X t as this Y t + Z t, which is equal to S N t + 1 - S N t. Yesterday, we were able to use essentially strong law of large numbers to work out the time average residual life, time average age and time average duration. For example, we proved that 1 over t, you send limit t tending to infinity integral 0 to t Y tau d tau is equal to limit t tending to infinity 1 over t 0 to t Z tau d tau.

Both are equal to second moment of the interarrival times over 2 X bar; this is almost surely. And we also showed, from this, this is clear that t tending to infinity 1 over t integral 0 to t X tau d tau is expected X square over expected X, almost surely. These are respectively the time average residual time, time average age and time average duration. So, these are time averages, right? So, later we will prove ensemble average version of this; namely that if you take a particular t, you take expected Y t; Y t is of course a random variable at any given time.

After a long time, expected Y t is equal to; these are ensemble averages; equal to what answer do you expect? You expect the same answer. We will prove this later, more involved. Now there is no almost surely, it is just numbers. And likewise, so, these are ensemble averages. So, these are time averages. So, for the ensemble average, you are taking a very large time t. So, Y t or Z t or X t is a random variable.

You look at a particular t; for example, X t is simply the duration S N t + 1 - S N t, the width of the interval that t is sitting in. The expected value of X t is of course a random variable for each t. See, X t is a random variable for each t, so, you take its expectation; and you send t to infinity; you get this number. So, this is ensemble average; you are averaging over various omegas, as opposed to averaging over time.

Now, I pointed out yesterday that these residual time, age, duration, the expected value of these quantities is directly proportional to the second moment of the interarrival times. So, if you have this X random variable; if it can take large values with some substantial probability, expected X square will become very large. So, typically, the duration or the age or the residual times will be very large.

In fact, if you look at this quantity here, this guy, which is the expected or the average duration, it is simply the width of the interval that t is sitting in. There is a bus process which is a renewal process, let us say; and you are showing up at some time t. And the interarrival time of the buses in which you arrive has expected value or time average expected X square over expected X.

Is this bigger than expected X or smaller than expected X? Or can you say something about this? See, what is the average inter-bus arrival time if you have this renewal process of bus arrivals? Expected X. Now, we are saying that, if you come to the bus stop at some time t, the time duration between the previous bus and the next bus you will see, has expected value or time average value in this case, expected X square over expected X.

My question is, is this bigger on an average than expected X, or smaller, or can you say something at all? So, how does this quantity compare to expected X? It is bigger, right? So, I am saying that this guy is bigger than; so, whatever is here, is bigger than or equal to expected X. Why? Variance is non-negative, or Jensen's inequality, X square is 1 X, right?

Anyway, so, the average duration of the interval in which you arrive to the bus stop is larger than a typical interarrival time.

That is because it is a special duration which contains your arrival, and it tends to be therefore larger, is statistically larger. So, these extraneous arrivals take place to the system when; are they more likely to take place to the system where the interarrival times are larger than usual? That is intuitively what we are saying. Is that clear? Now, in this lecture, we will generalise this sort of time average calculations to more general reward functions.

So, you can think of the residual time or the age or the duration to be some sort of a reward function associated with the renewal process. So, you can think of Y t for example. At each time t, you are accumulating a reward Y t and you are looking at the time average reward or Z t or X t for that matter. The only thing that is special about Y t or X t or Z t is that, these are random variables; for every t, Y t is a random variable.

These random variables depend only on things to do with that renewal duration. If you are at a time t, that reward Y of t or Z of t or whatever, is only a function of whatever is happening in that renewal duration. It does not depend on previous renewals and renewals to come after and all that. So, it turns out that everything we have done so far can be generalised to more general reward functions which depend only on things happening in that renewal interval.

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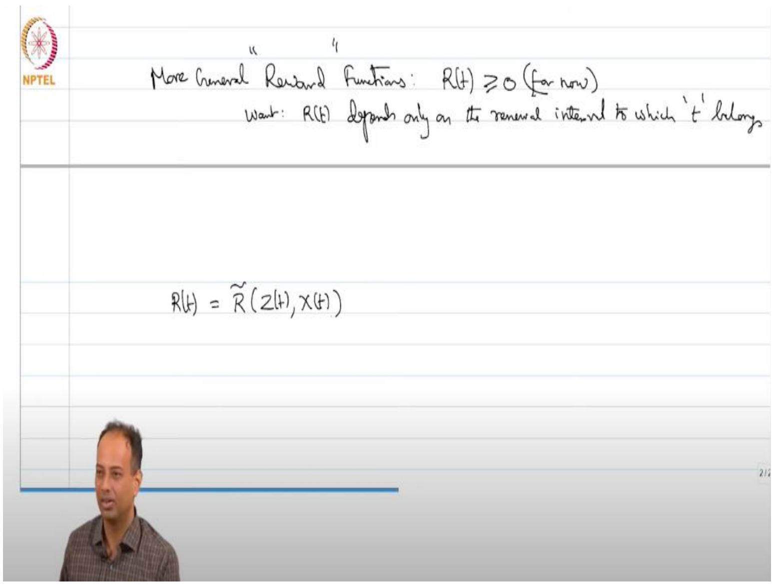

So, let us talk about more general reward functions. So, I want to talk about R of t, reward function. So, this is a reward function; let us say it is non-negative for now. It can be negative, but to keep things simple, let us keep positive rewards, no losses. And what I want is that, intuitively I want R of t depends only on the renewal interval to which t belongs. So, examples are, R of t can be Y of t, Z of t, X of t.

It can be or some other functions here off also; it can be Z square t or plus Y t, whatever; it can be anything that depends only on things happening in that renewal duration. So, in fact we can; the simplest kind of reward function is something that is; let us call this R tilde; something that depends on; it is only a function of Z t and X t, or Y t and X t; it does not matter, because X t is; you can write any one of these two.

I am just writing Z t, X t. So, this is one option. So, such an R of t would depend only on that particular renewal duration containing t. So, this kind of reward functions are certainly legitimate. I would like to stick to this form of reward function for now. We can in fact allow for reward functions which also depend on some independent source of randomness, which have nothing to do with the renewal process for example.

So, you could have coin tosses, you could have some other Poisson process which is independent of the renewal process, look at those Poisson arrivals during that renewal interval and all that. All that is allowed; we we will look at all that; but just to fix ideas, I want to look at R of t which is just a function of age and duration, or age and residual time, some function there off. So, this could be; so, many a nice function of Z square t plus X square t or whatever. So, this kind of a setting I want to calculate; so, this is legitimate; so, I want to calculate what?

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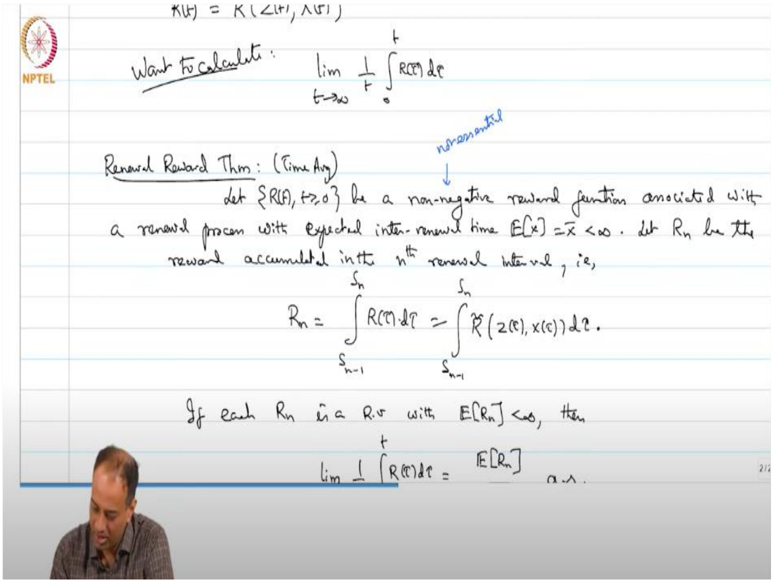

I want to calculate integral 0 to t R tau d tau. So, this would be R tau. What we have done so far is, we have taken R tau is equal to Y tau or Z tau or X tau; this is what I want. And the corresponding ensemble version would be expected R t, which we will do later. And you would expect that the answers to be the same. So, I want to state the main theorem which governs this time averages. This is called the renewal reward theorem, time average version.

There is also renewal reward theorem, ensemble average version. This says that, let R of t, t greater than or equal to 0 be a non-negative reward function associated with a renewal process with expected inter-renewal time, expectation of X is some X bar is finite. Let R n be the reward accumulated in the nth renewal interval, i.e. R n is equal to; it is simply integral.

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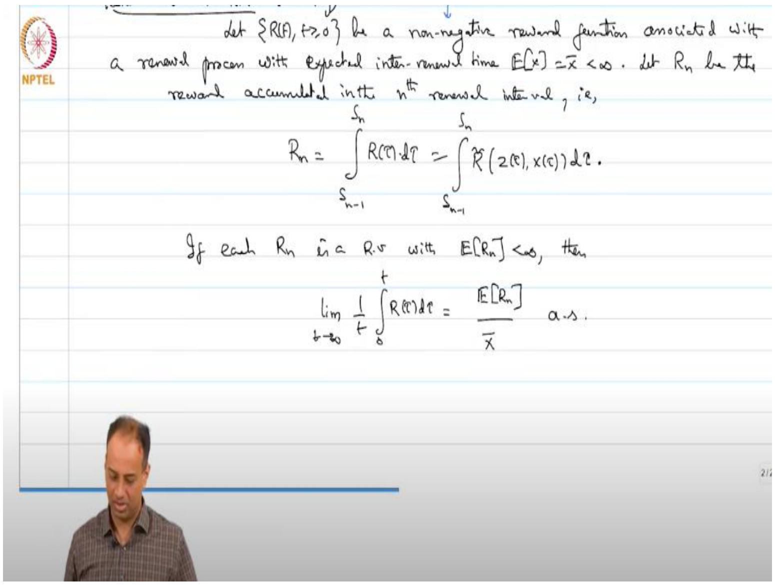

If each R n is a random variable with expected R n finite, then limit t tends to infinity 1 over t integral 0 to t R tau d tau equal to expected R n over X bar, almost surely. So, R of t is some reward function which is, we have; for fixing ideas, we will take as some function of Z t and X t. We are assuming non-negative; this is not essential; we can even accommodate losses, reward and losses. R of t can take negative values, but what you want is that, expected, absolute value of R n should be finite.

So, this non-negative is not such a key requirement. The key requirement is that, R of t should only depend on that; whatever the t is, look at that renewal interval, it should only depend on things happening in that renewal interval. We have fixed it to be R of Z t X t, but we can also later accommodate other independent randomness that is happening in that renewal interval, only in the renewal interval.

So, I want to calculate time average reward. What we are saying is that the time average reward is expected R n over X bar. Now, what is this expected R n? is the question. What is this R n? R n is a random variable which is obtained by integrating your reward function; R of t is your reward function, R of tau; integrating that reward function here over that particular renewal interval, nth renewal interval.

So, it is a random variable. See, S n and S n - 1 are all themselves random variables, so, R n is a random variable. And if R n is, expected value is finite, then the time average reward is equal to expected R n over X bar, almost surely. This is what the renewal reward theorem says. This is quite a powerful theorem and it can be used to analyse queueing systems among other things.

This Rn, so, for example, when we took R tau equal to, let us say, residual time, R n in that case was the area under the isosceles triangle; because, we drew those triangles, remember? And the reward accumulated in 1 renewal duration in that case was simply X n square over 2. So, this R n is generalising the concept of that area under that particular renewal interval. That is really all it is doing.

So, X n square; so, expected X square over 2 X bar you will get, if R is simply age. You will recover what we derived earlier. So, as you can imagine, the proof will be very similar; so, the upper bound and lower bound, and use strong law of large numbers.

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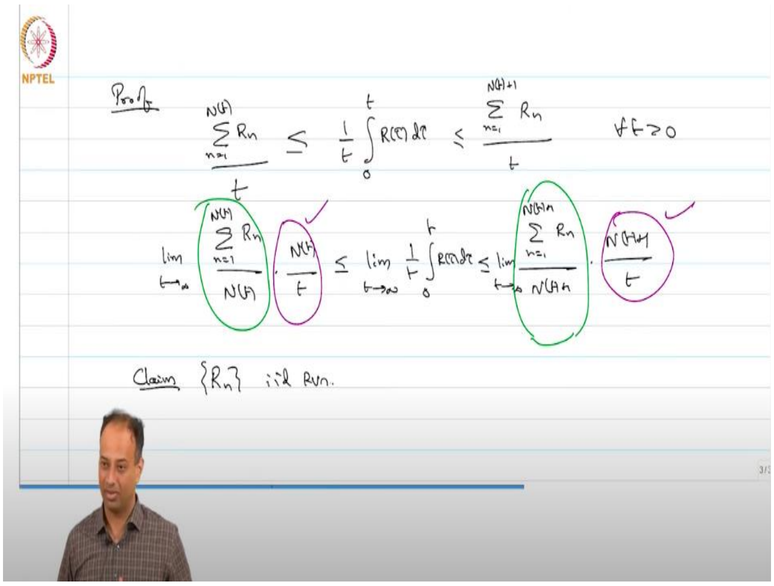

Proof: So, we can write like this; we can do the same trick. This is true for all t. Sum over n equals 1 to N t R n over t. I am basically doing the same trick. If you want to take 1 over t integral 0 to t, you are simply adding up; integral 0 to t R tau d tau is the sum of all the rewards obtained until t. That can also be calculated by adding all the rewards obtained in all the renewal intervals, until just before t and just after t.

That is what the upper bound, the lower bounds are. So, you look at t; you look at all the rewards accumulated till before the previous renewal; that is the lower bound; until after the, till just before the next interval; that is the upper bound. R n is the reward accumulated in each of these renewal intervals. In the previous special case, R n was simply X n square over 2, where the reward was just age. We are doing the same trick.

Now, we send t to infinity. And you have to do some N t over; you know, divide by N t, multiply by N t, that trick has to be replayed again. So, you can write limit t tend to infinity; R n over; n equals 1 to N t + 1 R n over N t + 1 times N t + 1 over t, limit. So, there should be a limit. Now, what do you think we should do? So, these guys, we already know how to handle. This guy we know; that guy we know.

These guys go to 1 over X bar, strong law for renewal processes. So, what remains is to handle that bit and that bit. So, it looks like a sum of bunch of these R n's divided by; it looks like a sample average of these R n's. Now, the key issue is that these R n's are IID. Claim is that; R n, remember, was X n square over 2 in the previous example; it has turned out to be, if X i's are IID, X i squares over 2 are IID. So, claim: these R n's are IID.

And that is because we have chosen the reward processs to be dependent only on the renewal interval. So, the reward accumulated over 1 renewal interval will be independent of the reward accumulated over the next renewal interval, and they will be identically distributed. To see this, you can actually just write it out.

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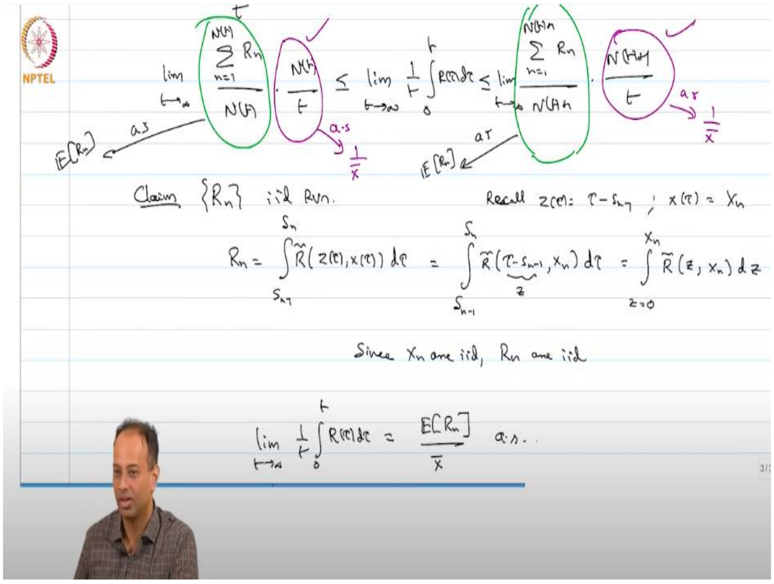

So, this R n is nothing but integral S n - 1 to S n R tilde; we have chosen this particular form, right; Z tau X tau d tau. So, this is equal to; so, remember that; recall, Z of tau is equal to tau minus S n - 1. I am going from S n - 1 to S n. So, Z of tau will be tau minus S n - 1. I am looking at the nth renewal interval. So, this will be integral S n - 1 to S n R tilde tau minus S n - 1; age will be S n - S n - 1, which is X n.

So, X tau; because this tau is sitting in S n - 1 to S n, so, X tau will simply be that particular X n, d tau, which you can rearrange by just a variable change to some Z is equal to 0 to X n R tilde Z, X n d tau. What have I done? I have put tau minus S n - 1 is equal to Z. So, now the Z integral will go from 0 to S n - S n - 1, which is integral 0 to X n. That is all that I have done. d, I have changed the variable of integration, have I not?

So, tau minus; so, this I am putting as Z, so, this should be d Z. It is just a simple variable change. So, Z is a variable of integration; so, R tilde is a function of Z and X n; and X get integrated out between 0 to X n. So, what comes out will only be a function of X n. So, this R n will be a function of X n, some f of X n. Like, in our previous example, R n was simply X n square over 2. In general, it is some function of X n. Therefore, R n's are IID. So, this shows that this is only a function of; so, this implies, since X n's are IID, R n's are IID.

So, going back; so, this will converge to what? This guy will almost surely converge to; see, N t increases to infinity, almost surely; so, this will converge to expectation of R n, which is just expectation of R. So, Rn's are IIDs, so, it will have some common expectation R. We can just write it as expectation of R n, no problem. It will not be a function of N. It will be a number. And this guy will converge to; so, this guy we know already, right?

This guy converges almost surely to 1 over X bar. This is also going to 1 over X bar, almost surely. This will go to the same limit. This also goes to expected R n or expected R. So, what I want to say is that; so, maybe I do not need such big an arrow. So, I should just say, this guy goes to, almost surely to expected R n. So, we can easily see therefore that limit t tending to infinity 1 over t integral 0 to t R tau d tau is almost surely equal to expected R n over expected X.

Same logic, nothing changes from what we did for residual time or age, except this R could be more general. Ideally, I would prefer to write expected R n as expected R. R n's are IID, so, they have a common expectation. But Gallager maintains the n, so, I am keeping the n. But please note that expected R n is common across all these n's; it will be a number. See, because the left side has no end, right? You think about it; left side is just a number.

For each realisation, you get a number of the process; each realisation of the stochastic process, you will get some number. So, the left-hand side is a random variable, really; but the left-hand side does not have n; so, the right-hand side really should not have n, and it does not. It appears that it has an n, but expected R n is the same for all n, because R n's are IID. So, the almost sure limit of this sequence of random variables is this number. That is what we are saying. Now, this has many applications.