Stochastic Modeling and the Theory of Queues

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Module - 4
Lecture - 26
Residual Life, Age and Duration (Time Average) - Part 2

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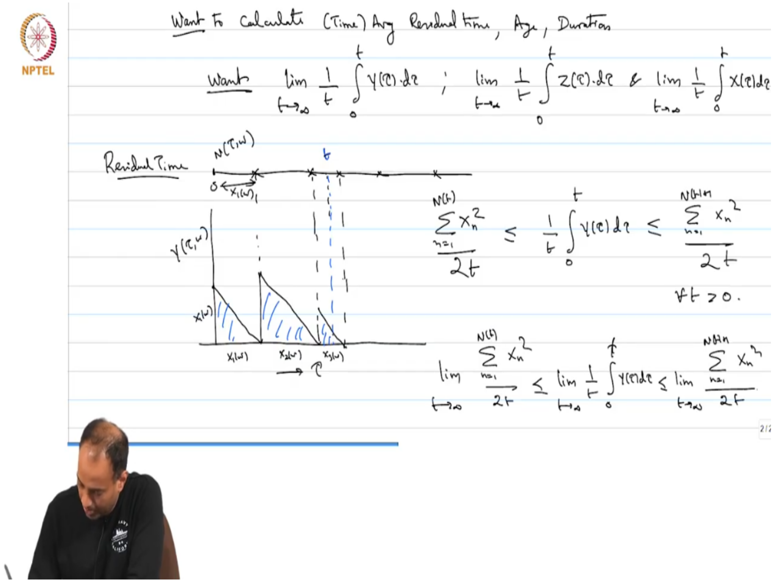

So, what do we want to do? Want to calculate average residual life; residual life, residual time are the same, okay? What did I say here? Residual life; which may be, say residual time.

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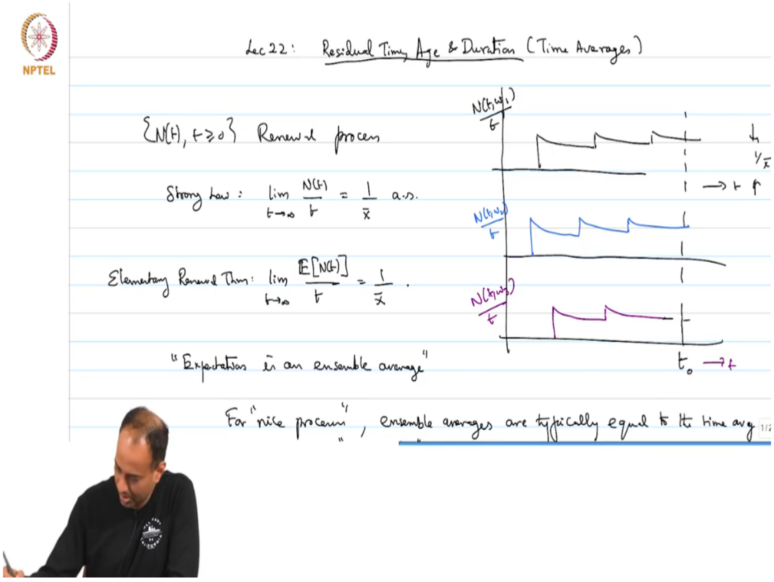

I want to calculate time average residual time, time average age and time average duration. Later, we will also calculate ensemble average residual life, ensemble average age, ensemble average duration; of course, the answers will be the same. So, at any given time t, I look at t and I see when is my next arrival; and that length is my Y of t. So, for every t, I have a Y t. So, I want time average residual life, residual time. So, how do I define that?

What is the time average of some function? If I give you some function f of t, what is its time average? Integrate it 1 over time, send time to infinity. That is what we will do. So, really, what we want to calculate is this. Want to calculate limit t tending to infinity 1 over t integral 0 to t Y tau d tau. Why did I write tau here? Because, I have t outside; I do not want the same variable. This is the time average of the function Y.

And likewise, I want to calculate limit t tending to infinity 1 over t integral 0 to t Z tau d tau, and limit t tends to infinity 1 over t integral 0 to t X tau d tau. This is what I want to calculate for a renewal process. Now, we can do this in pictures. Let us first consider residual time. Suppose my renewal process is running like this; can you please tell me what the plot of Y t will look like? So, this is time; I want to plot; or maybe I should call this Y tau.

So, I want to call this Y tau. So, at each tau; fix a time tau; I want to look at what the; I want to plot this function Y tau. So, maybe to be precise, I should call this; this is a particular omega, right? So, this is a particular realisation of the process, right? Let me call this; N t omega is realised like this; so, maybe I should call this, let us say this is N tau omega and this is Y tau omega, for that realisation of the process.

What is the residual time at time t = 0? It should be equal to this guy, X 1 of omega. So, here, it is equal to X 1 of omega. And then, at the first arrival what happens? It goes to 0, and it goes down like this, linearly. So, this is X 1 omega, and this is also X 1 omega. These are these isosceles triangle. And then again it goes to 0 here. Now, you got another; so, this renewal happened; now, what is the residual time?

You have to wait for the next renewal. So, that guy is X 2 of omega. So, suddenly the value jumps to X 2 of omega; and then, of course, decays down like that. You see what I mean; maybe I am not drawing it too well. So, this guy is nothing but X 2 omega and all that. Here, again you will get a X 3 omega and so on. So, you want to calculate the time average of this kind of a function. This is for a particular omega of course.

For a different little omega, you have a different set of interarrival times, you get a different plot of Y. So, you want to calculate this average, 1 over t 0 to t Y tau d tau. So, what we do is to write down this kind of a bound. So, you have this guy, right; so, let us say 1 over t integral 0 to t Y tau d tau; this is bounded, we can show, is bounded between 2 things. I am going to fill up what these 2 things are.

So, if you take some time; so, this is, let us say, a particular time, some time t. So, you want to calculate the total area under Y tau, till the time t. So, let us say, I fix a time t. You look at some; what is the total area integral 0 to t Y tau? So, I have to calculate the sum of all these guys. I have to calculate the area of all these guys. I want to bound that between the sum of all the areas of the triangles before that particular instant t, and lower bounded by just the sum of these two and upper bounded by the areas of sum of all these three.

So, what does that really translate to? It translates to something like this. Let me just write this down. So, this will be sum over n is equal to 1 to N t. What is the area of each of these triangles? Half X i square. So, I can write something like X n square. So, basically, I can write this. And sum over n equals 1 to N t + 1 X n square over 2; and then, I have to put a t, because there is a t here; there is a 1 by t outside the integral.

This is true for all t, just on the picture. So, I am fixing a t; I have to add up all areas till t. So, instead of just taking a part of this particular triangle, I am lower bounding it by sum of areas of all the previous triangles and upper bounding it by sum of all the triangles including the current one. So, X n square by 2 is simply the area of that particular triangle, isosceles triangle. It is a right angle isosceles triangle.

It is a very geometric, very pictorial thing, but this is certainly correct for all t. So, now, I have to send t to infinity. What I really want is, I want to put a limit t tending to infinity here. Then, if I send limit t tending to infinity here and here, in the lower bounds and upper bound, I will have a sandwiching; and hopefully the lower bound and upper bound go to the same thing. If they go to the same thing, I have a result for limit of whatever I want.

So, therefore, I can say, limit of t tending to infinity, sum over n equals 1 to N t X n square by 2t less than or equal to whatever I want, limit t tending to infinity 1 over t integral 0 to t Y tau d tau less than or equal to limit t tending to infinity, all this; n equals 1 to N t + 1 X n square over 2t. So, I will now do some manipulations.

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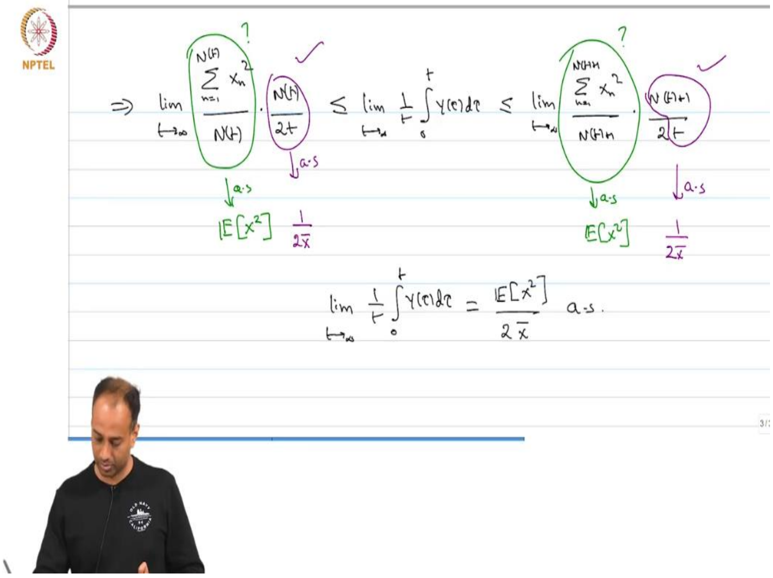

Therefore, so, what can I say? So, let me do the following. So, this I will write as limit t tending to infinity sum over n equals 1 to N t X n square. So, how many of these X n squares am I summing? N of t under the lower bound. So, what I want to do is to divide by N t, multiply by N t; so that they cancel; over 2t. I will just introduce, multiply it and divide it by N t. It is some non-zero, non-negative thing; I can do that.

So, less than or equal to whatever I want; limit t tends to infinity 1 over t integral 0 to t Y tau d tau, which is less than or equal to; I have to do another trick here, I have to do limit t tending to infinity. I have N t + 1 terms, so, now, what do I do? N t n equals 1 to N t + 1 X n square over N t + 1 times N t + 1 over 2t. So, I have just done some; I have drawn a picture; calculated these areas; upper bounded, lower bounded whatever I want; and then, just did some multiplying and dividing by N t.

Now, why did I do that? First of all, I get this familiar N t over t, which I know is 1 over X bar. This guy, N t over t is 1 over X bar as t tends to infinity. Then what is this term? If you look at; so, that guy; so, this term is familiar to me, N t over t is very familiar to me. Likewise, this N t + 1 over t also, this guy is also easily handled; that will also go to 1 over X bar. This both purple terms will go to 1 over X bar.

Question is, what happens to these guys? You have to use strong law of large numbers for X n square. See, we already know that N t goes to infinity as t goes to infinity. We proved, right? In great detail we proved that N t goes to infinity as t goes to infinity. So, limit t tends to infinity, this behaves like sum over some large number of X n square divided by the number of the terms. So, this is like a strong law for not X i but X i square.

X i square is also IID, right? If X i's are IID, X i squares are IID. So, this is like a sample average for X i squared, except the number of terms is also a random variable, N t; but we know that this random variable goes to infinity. So, this green term will converge to what? You would think that this should converge to expectation of what? X i square, which is just expectation of X square, second moment of these interarrival times, almost surely.

And this guy converges to; if I just take the whole thing, this thing converges to what? 1 over 2 X bar, because there is a 2 out there, almost surely. Now, again, there is one more result you have to invoke that if one sequence goes to expectation X square, another sequence goes to 1 over 2 X bar; the product sequence goes almost surely to the product of the limits. This you know from calculus real analysis, right?

If X n goes to X, Y n goes to Y, X n Y n goes to XY. But now, this is an almost sure statement, so, you have to look at the omegas where which the first convergence happens, the omegas for which the second convergence happens; intersect; then, you can get it; not a big deal. So, this guy will also go; so, if you look at the upper bound, this is also like a strong law; this will go almost surely to expectation of X square.

N t + 1 over t is not a problem, because 1 over t will go to 0. N t + 1 over t is just N t over t plus 1 over t; 1 over t is going to 0. So, this guy will go almost surely to 1 over 2 X bar. So, moral is, by sandwich theorem, we can say that limit t tending to infinity; this time average residual time is what? Expectation of X square over 2 X bar. See, what is in the numerator is not expectation of X, square; it is the expectation of, X square.

So, you are getting second moment over twice expectation of X. X bar is just expectation of X. So, this is your time average residual life. It depends on the second moment of the renewal distribution, the interarrival distribution, and this is of course, almost surely. So, you can try this for the Poisson for example. For the Poisson process, we know from memorylessness that the residual life is an exponential random variable; so, therefore, the average residual life should be 1 over lambda.

But if you plug into the formula, what do you get? What is the expectation of X square? Expectation of X square, the second moment of an exponential random variable with parameter lambda is 2 over lambda square. So, 2 over lambda square by 2 over lambda. So, you get 1 over lambda. So, it checks out. So, now, this formula is quite; you can apply to anything, right?

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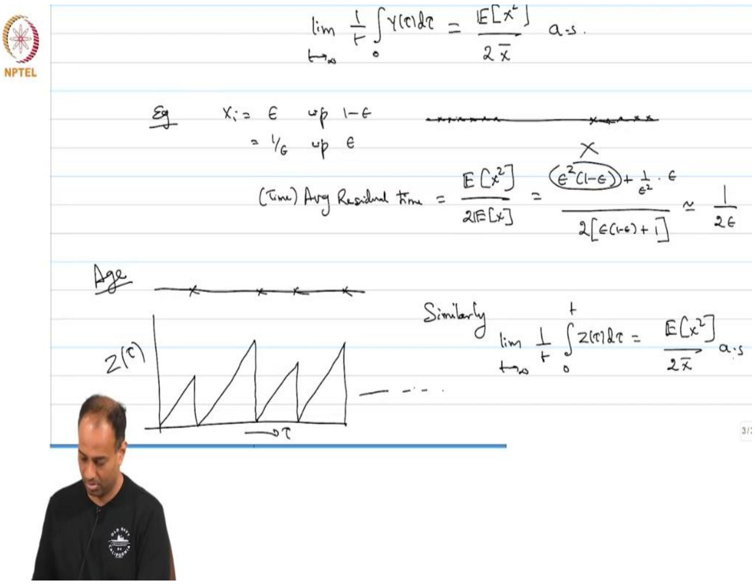

So, example, if you take the process that I mentioned, where with high probability you have arrivals which are very close, let us say, X i is equal to epsilon with probability, let us say, 1 minus epsilon, and is equal to 1 over epsilon with probability epsilon. So, epsilon is something very small. So, you get very close by arrivals with very high probability and one very long 1 over epsilon interarrival time with probability very small epsilon.

So, typically, the sample paths for this process will be like; you look at this process, it will have a bunch of very close arrivals. And then, for a very long time, you will not have any arrival. Again, you will have lots of arrivals. This process looks like this. Now, what is the average? Time average residual time is equal to expected X square by twice expected X, which is equal to; can you tell me please; what is expected X squared?

Plus 1 over epsilon square times epsilon over twice epsilon times 1 minus epsilon plus 1. I am just calculating the expectation in expectation squared. So, what is this roughly like? See, this guy is very small, right? So, epsilon is very small, this is very small; ignore epsilon square term. This this guy is roughly like 1 over epsilon. This also you ignore. So, this is roughly be like 1 over 2 epsilon.

So, the average residual time behaves like; it is pretty much determined by that 1 long interval; because, when you show up at that time t, you are very likely to be in that very long interval, intuitively speaking. This is just direct calculation. Likewise, you can also do age. What will the plot of age be like? We had plot of this sort of a thing for residual life. So, if this is your renewal process, age will look like that.

The isosceles triangles will be flipped; because, the farther away you go from a renewal, it grows larger, and then it falls to 0 when you see a; so, you get the idea. So, this is Z tau versus tau, for that particular omega. This is what the Z tau looks like. But actually, if you think about it, the areas are the same; it is just that the figure is reversed. So, similarly, you can argue that; same argument, except these triangles are; the areas will be the same; they just flipped.

Limit t tending to infinity 1 over t integral 0 to t Z tau d tau will be equal to same expected X square by 2 X bar, almost surely; symmetric consideration. Finally, you want to deal with duration.

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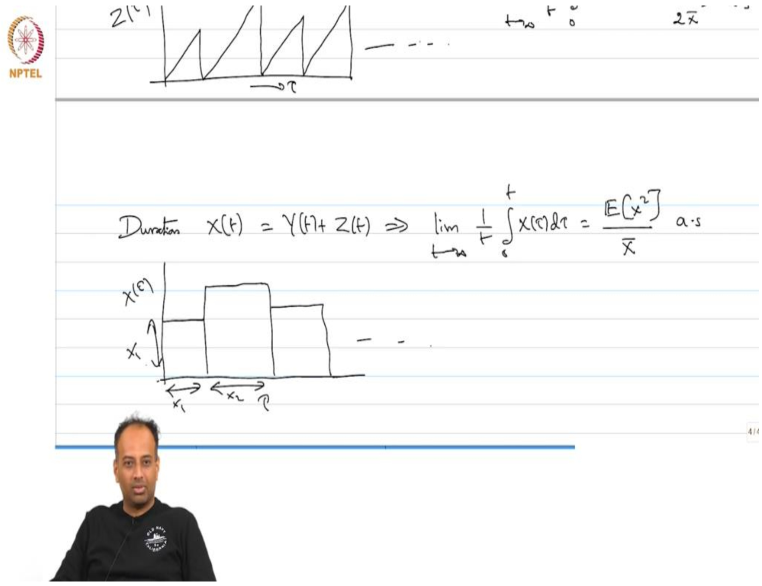

See, one way to deal with duration is just write it as Y t + Z t; so, from which you can get limit t tending to infinity 1 over t integral 0 to t X tau d tau is simply sum of those two, right?

The integrals are; so, you will get expectation of X square over X bar, almost surely; the 2 will go away. If you insist on drawing pictures, what will you do? Well, for the same process, if I were to draw X tau versus tau, you will have these square sort of; it will be constant, as Praful pointed out, right; between 2 arrivals, the duration is the same.

You see what I mean, right? You will get something like that. And of course, that is where; so, this will be X 2, this will be X 1 and so on. So, this will also be X 1. So, you are adding the areas of squares instead of isosceles triangles. It will be X i squared; you get 1 over X bar from strong law. That is how this comes about. So, what does this tell you? Moral: So, if you show up at a particular time, the average duration of the buses, the duration between the previous bus and the next bus that you are going to see is dependent on the second moment of the interarrival times.

And there are distributions for which the second moment can be very large. For example; the example I gave you, right? The second moment is like 1 over epsilon, although the expected X is roughly like 1. So, the average interarrival time is roughly 1, but the second moment is like 1 by epsilon. So, the duration in which you show up is extraordinarily large, in this sort of a distribution. Is that clear? So, this is a good introduction to this sampling paradox as it is called.