Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology - Madras

Module - 4
Lecture - 25
Residual Life, Age and Duration (Time Average) - Part 1

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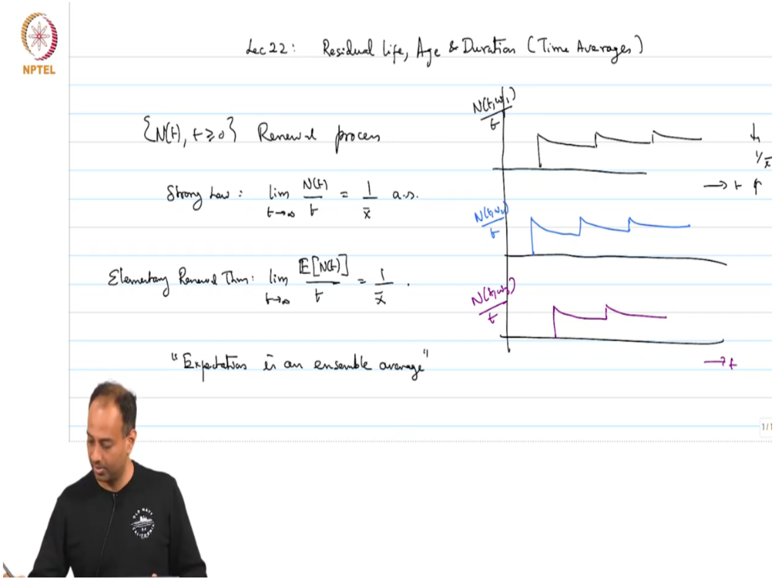

Welcome back. Today we will discuss about residual life, age and duration of a renewal process, only restricting to time averages. So, before we get into this, I would just like to refresh your memory on the difference between the time averages and ensemble averages. Just recall that N t is a renewal process. We looked at strong law, last class, this is where we left off.

Strong law says that limit t tending to infinity N t over t is equal to 1 over X bar, almost surely. So, this strong law is an example of a time average statement. So, just to draw some pictures for you; so, let us say this is N t over t. So, N t, let us say for a particular omega 1 over t against t. So, this will look like; so, this horizontal axis is time. It will look like jump, decay, jump, decay, jump, decay and so on.

So, wherever you have an arrival, the N t over t jumps by 1 over t, whatever the time is. And then it decays until the next arrival; there is a smaller jump and so on. So, what we are saying is that, as t becomes very large, this sample path N t of omega 1 over t settles down at 1 over

X bar for almost all omega. It means, on a set of little omegas whose probability is 1, we have N t omega over t converging to 1 over X bar.

So, for a very large values of t, you will see that this converges to 1 over X bar. So, if you look at a different sample path, let us say some omega 2. So, this is, let us say N t omega 2 over t. You will have a different sort of a; I mean, qualitatively same kind of a sample path, but it depends on where the arrivals occur. You will have something like this. And then, let us say for another omega 3, you have something else. You may have; and so on.

So, this is for N t omega 3 over t or versus t. So, what we are saying is that the set of little omegas where these functions N t little omega over t does not converge to 1 over X bar, has probability 0. On a set of probability 1, we have convergence to 1 over X bar. So, this is what we mean by almost sure convergence, and this is what the strong law says. So, this is a typical example of a time average.

You take average over time, N t over t, the average arrival rate and you say that this goes over almost all the sample path to 1 over X bar. There is another very important theorem which is known as the elementary renewal theorem. It says that, limit t tending to infinity expected N t over t is equal to 1 over X bar. So, this is an example. So, this expectation of course can be; I mean, you can put the t inside the expectation, because t is not a random variable.

So, you can write it as expectation of N t over t; nothing wrong with that. By the way, so, this limit is not almost surely or anything; it is like expectation of N t over t is a; for each t, it is a number. So, it is just convergence of some sequence of numbers; the sequence is indexed by t, that is all. So, I want to say that this is, in terms of meaning it is very different; that I want to first convey.

The second thing I want to tell you straight away is that the elementary renewal theorem is in no way a consequence of strong law. It looks as though; I mean, what is the big deal? It looks like if N t over t goes almost surely to 1 over X bar, the expectation, you might think should go to 1 over X bar. It is absolutely not true. If your sequence of random variables converges almost surely to X, it is absolutely not true that expectations converge.

And in any case, proving the elementary renewal theorem is much harder than proving the strong law. We need to build up some more machineries to do that. So, the two things I want you to remember now is that elementary renewal theorem is a completely different result, no way a consequence of strong law. Second, I want you to understand that it means something different from strong law. Strong law is just time averages, as I have drawn here.

So, when you take expectation; so, you can think about it like this. So, informally you can say that expectation is an ensemble average. The ensemble average, we mean we are averaging over various realisations of the process. So, you look at, let us say you fix a large t.

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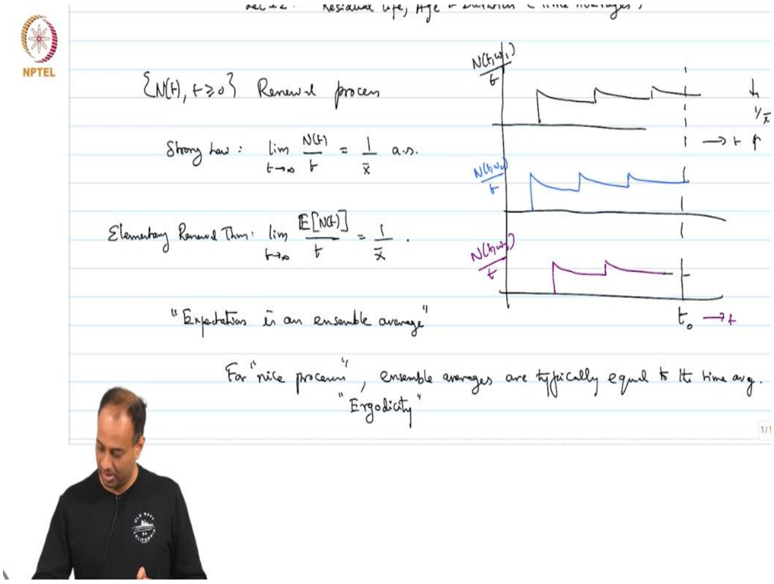

Let us say you fix some particular time; let us call this t nought. Let us fix a time t nought, let us say, which is large. You are looking at expectation of N t nought over t for large t nought, in the elementary renewal theorem. So, what you are doing is that, for fixing a t nought, you are looking at N t nought omega 1 over t nought, N t nought omega 2 over t nought and so on. And these different omegas have different probabilities, so to speak, as determined by the underlying distribution of these X's; because little omega, what is this little omega?

Once the little omega realises, all your X i's release, all your S i's realise. So, depending on the realisation of your interarrival times, you will have some particular realisation of N t omega. And these little omegas have different probabilities so to speak. And expectation of N t is like averaging over little omega, somewhat loosely speaking. And for a particular t nought, N t over t is a random variable; you are taking expectation over that.

You are saying that, if t nought is large enough; so, you first take expectation. You fix a t nought; take expectation of N t nought over t; there will be a number, for each t nought. Now, now you send t nought to infinity. So, that limit is 1 over X bar. That is what the element renewal theorem says. So, there is a t tending to infinity of course, but it is first an ensemble average. It is an ensemble average; there is an expectation first.

So, that is a different kind of result. And of course, the answers are the same; the ensemble average equal to time average. So, this property for nice processes, ensemble averages are typically equal to the time average. And this property is known as ergodicity. So, you see already that strong law is a time average result; it says time average converges to 1 over X bar almost surely.

Elementary renewal theorem is an ensemble result; you are taking ensemble average. So, there you are saying that the ensemble average is also 1 over X bar. So, it is some sort of an ergodicity that you see here, that the time average is equal to ensemble average. And we will repeatedly see this. You will take time average; you take ensemble average; often for renewal processes or Markov chains under certain reasonable conditions, you will get time average is equal to ensemble average, numerically, but the interpretations are very different.

So, time average means something very different from the ensemble average; but, if for nice processes such as these renewal processes and Markov chains which satisfy certain conditions, we will have them to be numerically equal, the time averages and the ensemble averages. This is a recurring theme in this course; so, I want to already point this out at the first instance when we see this to be equal; of course, they are two very different results.

I am not saying that because of ergodicity they are equal; I am saying that ergodicity follows from strong law and elementary renewal theorem, which are two big results by themselves. And hence, we conclude that there is a certain ergodicity in the renewal process. Is that clear? So, that is just a prelude. What I wanted to discuss today was residual life, age and duration. And you will discuss time averages; I will tell you what residual life is; I will tell you what age is; I will tell you what duration is; and I will help you calculate the time average residual life, time average age and time average duration.

And you will not be surprised to see that when we do ensemble averages, much later, the answers will be the same, but the reason they are equal will be more non-trivial. We will use reward theory to calculate both time averages and ensemble averages. So, I will tell you what residual time is.

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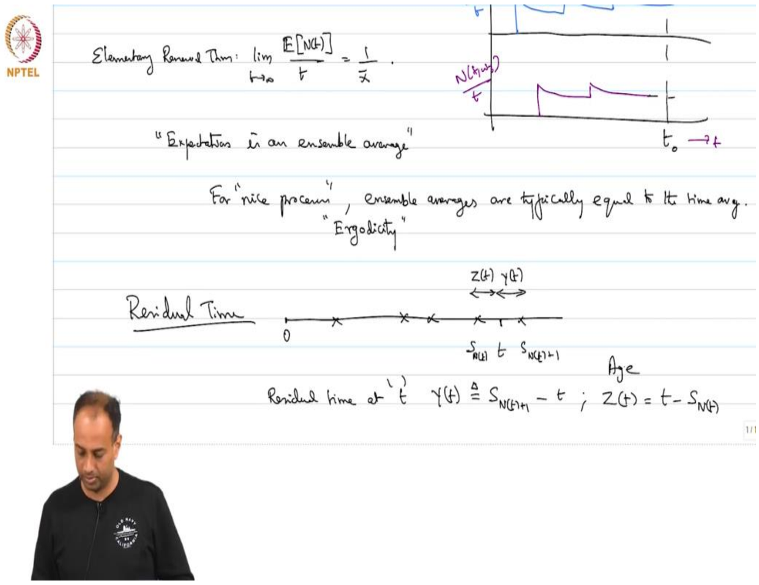

So, residual time: What is the residual time? So, let us say you have this renewal process; it starts at 0. These X i's are IID; you know what a renewal process is; and you take a particular time t. What is the time epoch of the previous arrival? S N t. See, at t, there have been N t arrivals. So, if you look back from t, the N t'th arrival, whenever it took place, the epoch is nothing but S N t. And likewise, this epoch is S N t + 1.

Now, I will tell you what residual time is. Residual time at t; so, you are looking at a particular t and defining the residual time at this particular time t. It is defined as S n; so, Y t is the usual notation that is used in Gallagher's book; is simply S N t + 1 - t. So, Y t is this guy. So, I am fixing a t; I am looking at the arrival before it and the arrival after it. So, Y t, the residual time at time t is the time to wait for the next arrival to come after t.

You are looking at a t, and the time to the next arrival is the residual time. So, in a physical example, so, this renewal process may be arrival of buses to a bus stop; you showed up at some time t; how long will you wait for the next bus? That is the residual time. And analogously, age is just t - S N t. So, you showed up at the bus stop; how long ago was the previous bus? So, this length is called Z t, age. This is age, Z t. So, it is called age because, how old is the previous renewal; how long ago was the previous renewal?

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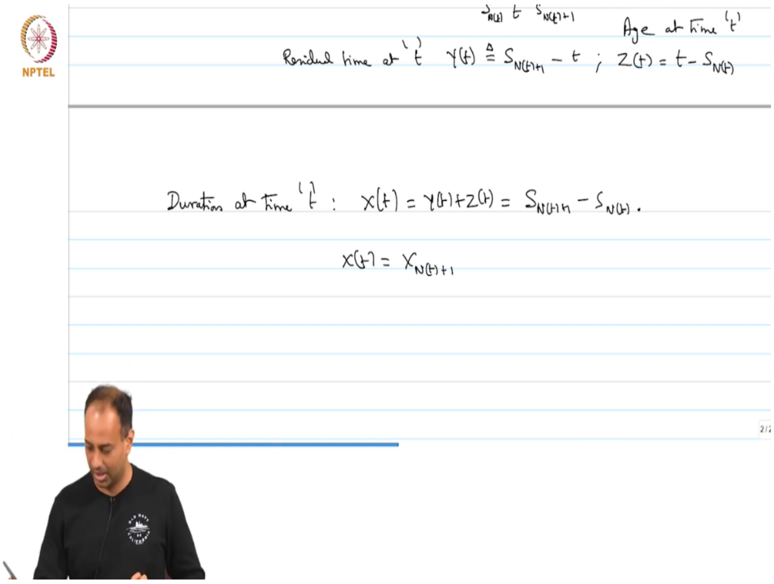

And duration; so, this is age at time t; and duration at time t is nothing but; so, let us denote it as X of t which is basically just Y t + Z t, which is equal to S N t + 1 - S N t. So, X t is Z t + Y t; the age plus residual life is duration. So, you are showing up at the bus stop at time t; how long is the duration between the previous bus and the bus you are going to see next? That is the duration. Any questions on this?

"Professor - student conversation starts" Duration; yeah, you are right. If it so happens that there are no arrivals between t and t prime, then, duration at time t and duration at time t prime will be the same. Of course, the residual life and age will be different. If I choose some other t prime here, then the duration will be same but not; age will be shorter and the residual time will be longer, or the other way around; assuming there is no arrivals between the two times.

But for a given t, this is clearly defined. Y of t is S N t + 1 - t; Z of t is t - S N t; and X of t is S N t + 1 - S N t; simple as that. **"Professor - student conversation ends"** So, in this, one of the things we will do is to calculate things like, what is your average residual life? what is your average age? what is your average duration? Well, the average could be time average or ensemble average; of course they will turn out to be numerically equal, but the meanings are very different.

So, we will calculate the average residual time, average age, average duration, time average; and the answers will be the same for the expected value as well, the ensemble average as well. So, one thing I want to point out, and this will become more obvious later, is that this duration at time t, which is S N t + 1 - S N t, is not your typical renewal duration. Is that somewhat clear to you at this point?

So, what I am saying is that; so, what is this S N t + 1 - S N t is that, is also basically this guy, right? So, X of t is nothing but X N t + 1, by where X i's are the IID random variables which define your renewal process; but this X of t which is X N t + 1, it does not have the same distribution as your typical X i intuitively. Why? It is a very special interval which contains t. So, it is not; we will see in fact that, we will derive a full characterisation of the distribution of X t's and Y t's and Z t's; but already it should be, you should get used to the idea that the interval in which you show up is an unusual interval.

The buses are running like a renewal process, let us say, at a bus stop. You show up at the bus stop at some particular time. And because it contains your arrival; you are some exogenous person coming to that bus stop. That duration between the previous bus arrival and next bus arrival is not typical. So, for example, in a Poisson process, let us say; this is the most concrete example that we know of; Poisson process is of course a renewal process.

So, you show up at time t. We know from memorylessness of the Poisson process that the residual time is exponentially distributed. So, Y t is an exponential with parameter lambda, whatever the process parameter is. And likewise, if you run the process in backward time, also it looks like a Poisson process. So, if you look at t, the time since the last arrival, which is the age, is also exponentially distributed.

And because of this memorylessness, you can argue that this Y t and Z t are independent for the Poisson process. Generally not true, but Y t and Z t are independent for the Poisson process. So, Y t and Z t are both exponentially distributed with parameter lambda, and they are independent. So, the entire duration between the previous arrival and the next arrival is distributed like a Erlang 2; sum of 2 independent exponentials.

So, even in a Poisson process, you are already seeing this. You have a Poisson process running; all interarrival times are exponential with parameter lambda. But you fix a t and you look at the time between the arrival before t and the arrival after t, it is not exponentially distributed; it is an Erlang 2 distribution. In fact, the expected value of that time will be how much? Twice as large as a typical duration; so, typical interarrival time.

The duration is twice as long in expectation, for the Poisson process. So, you are already beginning to see that the duration is not a typical renewal time. So, we want to know what is its distribution? You see what I mean? It is a very special interval which contains t; it contains you are arriving at the bus stop. So, you do not know anything about the bus process, right? You just showed up at some time t.

So, just to drive home this point, let me give you another example which may be slightly pathological. Suppose your bus interarrival times are very small with very high probability, let us say epsilon; and very high with small probability. So, you have a bunch of busses coming very close together; and then, there is a huge gap. In this kind of a situation, the time t that you pick is very likely to be in the; see what I mean, right; if you just show up at this process, you are much very likely to show up at one of these long interarrival times.

So, although the expected whatever, the distribution of the time between 2 buses is very small, typically; because, with very high probability, you get 2 buses very close together. The interval in which you showed up to the bus stop is very likely to be very large, which is probably why you feel that whenever you show up to a bus stop, you do not get a bus for a long time. And in such a duration, a lot of people show up to the bus stop; so, the bus in which you go, tends to be very crowded.

This is known as sampling paradox; it is known by various names; one name is sampling paradox. The bus in which you travel tends to be very crowded. Likewise, even in social networks, you see this kind of a phenomenon. You are likely to run into nodes with lots of friends and all that. So, anyway, I just wanted to point out that this is a phenomenon that we will study in detail. And this X N t + 1 which is the duration is an atypical time between 2 consecutive arrivals.