Stochastic Modeling and the Theory of Queues
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Module - 3
Lecture - 23
Strong Law for Renewal Processes

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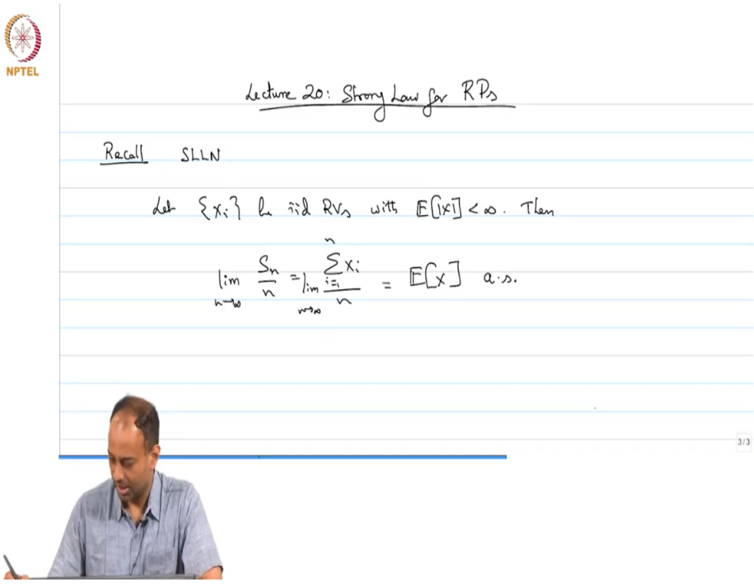

Strong Law for Renewal Processes: Recall strong law of large numbers. What does it say? Let $\{X_i\}$ be IID random variables, sequence of IID random variables with $E[X] < \infty$. So, you have,

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} = E[X]$$ almost surely. This is the strong law of large numbers for IID random variables with finite mean.

This is what strong law says. So, if you take, in a renewal process, $\{X_i\}$ are just some non-negative, well, actually positive IID random variables, and S_n corresponds to the n^{th} arrival epoch. So, if you take the n^{th} arrival epoch, divide by n and send n to ∞, you will get $E[X]$ almost surely; it is a straight consequence of the strong law of large numbers. Now, the strong law for renewal processes is written not for S_n, but for $N(t)$. So, let me write down.

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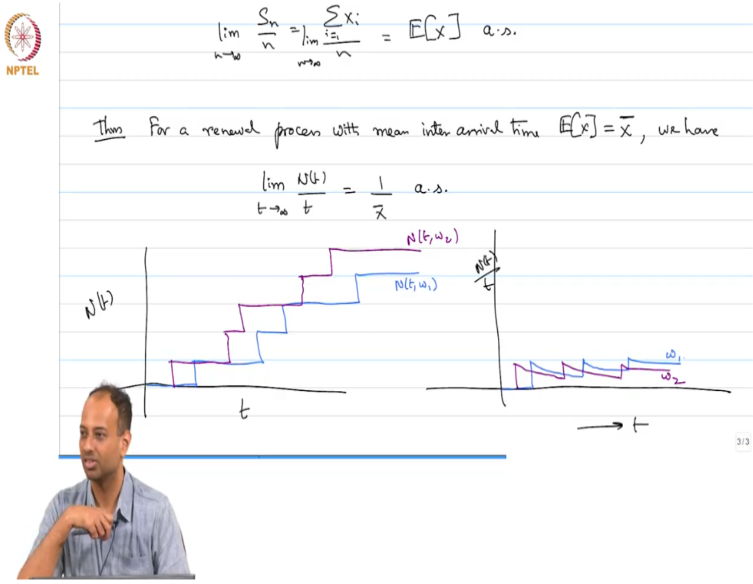

Theorem: For a renewal process with mean interarrival time $E[X] = \bar{X}$, we have $\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\bar{X}}$ almost surely, and this is true even if $\bar{X} = \infty$. So, for the strong law, the plain strong law for IID random variables, you need a finite mean, for $\frac{S_n}{n}$ to converge to $E[X]$ almost surely.

For the renewal process, $\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\bar{X}}$ almost surely, regardless of whether $\bar{X} = \infty$ or $\bar{X} < \infty$. We will only be bothered with the case where $\bar{X} < \infty$. This is true even for $\bar{X} = \infty$. And the way you prove that is, prove it for the case when $\bar{X} < \infty$, then use a truncation argument. So, our job is to prove this theorem. Let us first pictorially see what this is saying.

So, recall that, if you look at a plot of $N(t)$ versus t, you will get some kind of a step; at every arrival you have, the process steps up by 1. So, this is, let us say $N(t, \omega_1)$. For some other value of ω, you could have a different step function. You will have; this is a process $N(t, \omega_2)$. This is the different sample path of the process. So, you are taking $N(t)$ over t, which means that for each sample path, you are looking at $N(t, \omega)$ over t.

So, for each fixed ω, $N(t, \omega)$ is some function of time, is a step function. You take $N(t, \omega)$ over t. What we are saying is that, if you send t to ∞, $\frac{N(t,\omega)}{t} \rightarrow \frac{1}{\bar{X}}$ for a set of ω lying on a set of probability 1. On a set of probability 1, $\frac{N(t,\omega)}{t} \rightarrow \frac{1}{\bar{X}}$. And the set of ω's where $\frac{N(t,\omega)}{t}$ does not go to $\frac{1}{\bar{X}}$, does not converge at all, or converges to something other than $\frac{1}{\bar{X}}$; has total probability 0.

That is what this is saying. To be more explicit, if you want, you can plot the $\frac{N(t,\omega)}{t}$ as a function of time. So, if you do that; so, how is $\frac{N(t)}{t}$ going to look? The first arrival occurs, is going to jump; and then it is going to decay; then it is going to jump again; again going to decay; it is going to keep doing that; but the value of the jump is going to get smaller and smaller. Why? Because you are going to increase t.

So, every time there is an arrival, this $N(t)$; so, I am here plotting not $N(t)$, but $\frac{N(t)}{t}$, over, It is clear, no? I do not think I drew very well. But every time you get an arrival, you get a jump; but the height of the jump will be, not 1, but how much? $\frac{1}{t}$, where t is the time at which the arrival happens. So, this, what we are saying is that, as time becomes very large, the height which this guy settles at is $\frac{1}{\bar{X}}$.

So, in a different sample path, you could have; and so on; So, the purple guy is ω_2; the blue guy is some ω_1. What we are saying is that, this function which jumps up and then comes down, jumps up and comes down, this guy will settle at $\frac{1}{\bar{X}}$ for almost all ω, meaning that, in a set of probability 1, this function will converge to $\frac{1}{\bar{X}}$. That is the intuitive meaning of what this strong law of large numbers is saying.

Is that clear? So, this $\frac{1}{\bar{X}}$ can be thought of as the rate of a renewal process. So, $\frac{N(t)}{t}$ is the total number of arrivals in time t, per unit time, because $N(t)$ is the total number of arrivals in $(0, t]$; you are dividing by total time t; so, $\frac{N(t)}{t}$ is the number of arrivals per unit time. And as t becomes large, this is equal to 1 over interarrival time. So, this 1 over expected interarrival time is the rate of the process.

So, $\frac{1}{\bar{X}}$ has the interpretation of the rate of the process. For a Poisson process, of course, this is true, because, well, I mean, Poisson process is just a renewal process; and $\frac{1}{\bar{X}}$ there is just $\frac{1}{\lambda}$, which is λ, which is the rate of the process.

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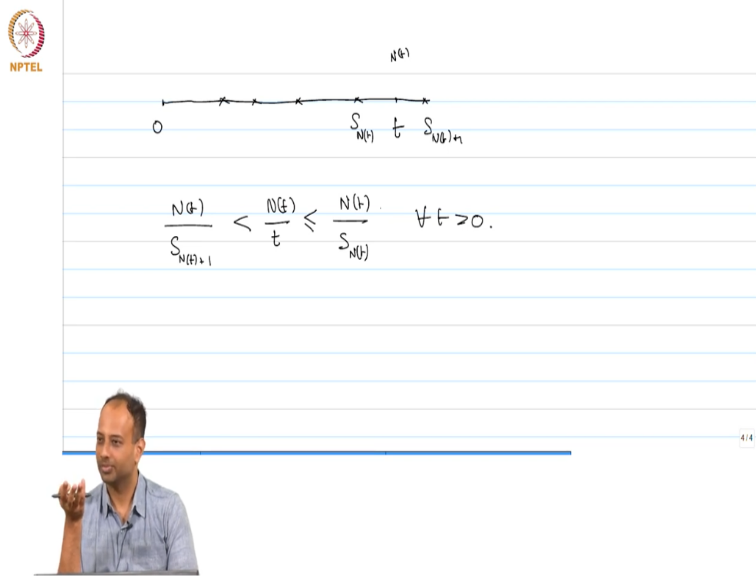

So, let us try and prove this. So, let me draw this picture. So, this renewal process is running, 0; so, let us say this is my t. So, at time t, there have been $N(t)$ arrivals. Now, can someone tell me what is the time of arrival of the most recent time; the most recent customer arrival, what is the time of that arrival? So, $N(t)$ is the number of arrivals that have come, so, what is the arrival epoch of the most recent arrival? Yes? What is n?

So, if $N(t)$ is a random variable, so, what is the time of the arrival which came, epoch of the arrival that came the most recently? So, at t, there have been $N(t)$ arrivals, so, when did the previous arrival come? $S_{N(t)}$ is the epoch of the most recent arrival. No, it is not clear? See, suppose $N(t) = n$, so, there have been n arrivals; you look back to see when the most recent arrival came; so, it is the S_n, right?

So, $S_{N(t)}$ is the epoch of the most recent arrival. Likewise, what is the epoch of the arrival that is going to come next? $S_{N(t)+1}$. So, this much is clear, right? So, I just want to build up an intuitive argument first. So, if you look at this $\frac{N(t)}{t}$, I can write this as being sandwiched between; let me see; so, can I write this: $\frac{N(t)}{S_{N(t)+1}} < \frac{N(t)}{t} \leq \frac{N(t)}{S_{N(t)}}$

Is this correct? I am making the denominator smaller, so that this ratio can get only bigger.

Likewise, I can write $\frac{N(t)}{S_{N(t)+1}}$; is it correct? So, you will agree. So, now I want to make t very large. If I send t very large, I want to see what happens to $\frac{N(t)}{t}$. So, this is true for all $t \geq 0$, let us say all $t > 0$. So, if I send t very large, this $\frac{N(t)}{t}$, which is the quantity of interest for me, is going to remain sandwiched between these two things.

So, if I want to prove that $\frac{N(t)}{t}$ goes to some limit, it is enough to show that the thing that is sandwiched between, is also going to that limit. Then I am done, right; by sandwich theorem, I would have finished showing what I want to show. Now, let us look at this object and that object. So, let us look at this. As t becomes large, what does this behave like? So, first, when the first arrival comes, this will be $\frac{1}{S_1}$.

Then two arrivals will become $\frac{2}{S_2}$. So, if you have little n arrivals, it will become $\frac{n}{S_n}$; but as t becomes larger and larger, there will be more and more arrivals, and $N(t)$ will go through all the positive integers. So, this should behave like what? $\frac{N(t)}{S_{N(t)}}$ will behave like $\frac{n}{S_n}$, where n is going to ∞. So, intuitively, $\frac{n}{S_n}$ should converge to what? Say $\frac{S_n}{n} \rightarrow \bar{X}$ by strong law, so, $\frac{n}{S_n}$ should converge to $\frac{1}{\bar{X}}$.

So, I have told you some two, three things which I have not fully, rigorously justified, right? First is that, if $\frac{S_n}{n} \rightarrow \bar{X}$, should $\frac{n}{S_n} \rightarrow \frac{1}{\bar{X}}$?. You have to prove that. It is true, but you have to prove it. The other is that, as $t \rightarrow \infty$, you need $N(t)$ to be getting larger and larger; it has to go through all the integers and go to ∞.

So, first of all, you have to prove that, as $t \rightarrow \infty$, $N(t) \rightarrow \infty$ almost surely. So, if I do those two things, I will be done. So, we need a couple of lemmas here; and then we use these two lemmas; the first lemma being that $N(t) \rightarrow \infty$ as $t \rightarrow \infty$; the other is that $\frac{n}{S_n} \rightarrow \frac{1}{\bar{X}}$ almost surely. So, actually, it is true for any continuous function.

If $X_n \rightarrow \alpha$ almost surely, then $f(X_n) \rightarrow f(\alpha)$ almost surely, for all continuous functions f. That we can prove. And the reciprocal function is a continuous function. So, that is all that, these two ingredients we need to prove this.