Stochastic Modeling and the Theory of Queues
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Module - 3
Lecture - 22
Introduction to Renewal Processes

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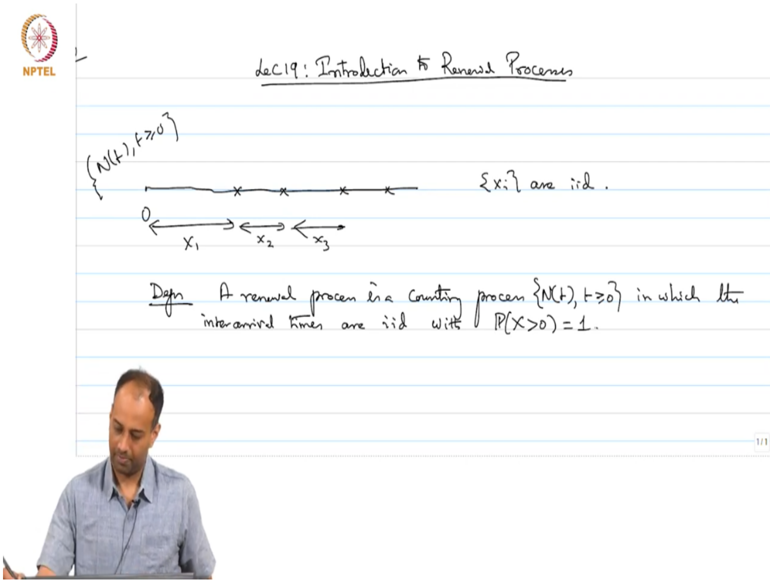

Good morning. Today, we begin our discussion of the next chapter, which is about renewal processes and renewal theory. So, we already know what a renewal process is; that is the good part; we already defined it. So, renewal process is a counting process $\{N(t),\ t \geq 0\}$; so, the renewal process is characterised by certain interarrival times $X_1, X_2, X_3, ..$ and so on, where $\{X_i\}$ are assumed to be independent and identically distributed.

That is what the renewal process is. So, definition: A renewal process; I think we already said this, I am just recalling; is a counting process $\{N(t),\ t \geq 0\}$ in which the interarrival times are independent and identically distributed, according to some distribution, underlying distribution X; I should say this; with $P(X > 0) = 1$. So, these $\{X_i\}$ are IID and the $P(X_i = 0) = 0$, which means that you do not get two arrivals at the same time with positive probability. So, this we assume. So, this is what a renewal process is, and we want to study $N(t)$, various things about $N(t)$.

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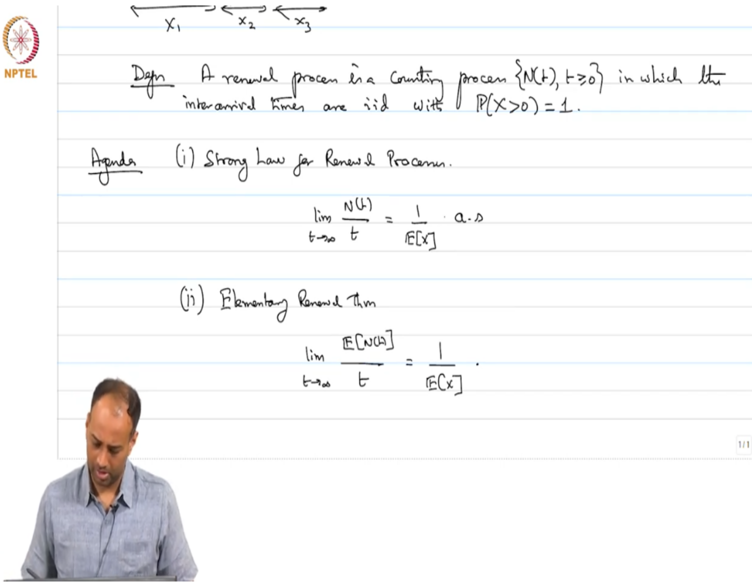

So, roughly our agenda in this chapter will be as follows: So, we will first prove a strong law for the renewal process. So, these $\{X_i\}$ are IID, and therefore, they satisfy a strong law. What is the strong law for $\{X_i\}$? If $\frac{1}{n}\sum_{i=1}^{n} X_i \rightarrow E[X]$ almost surely. So, something similar; you would expect that because this renewal process consists of these IID interarrival times, there should be a corresponding strong law for $N(t)$ as well.

And there is such a law. So, basically it says that,

$$\lim_{t\rightarrow\infty} \frac{N(t)}{t} = \frac{1}{E[X]} \text{ almost surely.}$$

Since X is always positive, $E[X]$ is always positive. So, this is what the strong law for renewal processes says. Now, as it happens, this $E[X]$ could actually be even ∞, but it is still okay.

The strong law for renewal processes holds regardless of whether $E[X]$ is finite or infinite. We will mostly study the $E[X]$ being finite, but even with $E[X]$ being infinite, this result still holds. So, in that case, $\lim_{t\rightarrow\infty} \frac{N(t)}{t} = 0$, almost surely, if $E[X]$ is infinite. Is it clear what this says? We are not going to prove this right now; we will prove it a little bit later.

Then, another very important result we will show about renewal processes is known as elementary renewal theorem, which says that

$$\lim_{t \to \infty} \frac{E[N(t)]}{t} = \frac{1}{E[X]}$$

So, the first is a time average; the second involves sending t to ∞, but also involves the taking $E[N(t)]$; it is an ensemble average. The answer is the same, $\frac{1}{E[X]}$.

And you might think that if $\frac{N(t)}{t}$ converges almost surely to $\frac{1}{E[X]}$, what is the big deal in saying that the $\frac{E[N(t)]}{t}$ converges to $\frac{1}{E[X]}$? Well, intuitively, of course, you would expect this, I mean, at least, you will not be shocked that this is true, but it turns out that the elementary renewal theorem is a very different statement, it is not in any way a direct consequence of strong law for renewal processes.

In fact, the elementary renewal theorem, the proof is not very elementary. Strong law proof is just a repetition of the strong law for large numbers with a few other tweaks. Elementary renewal theorem is a completely different result. And why it is a different result; what is the meaning of the elementary renewal theorem in contrast to the strong law; all that we will see. At a high level, strong law is a time average, elementary renewal theorem is an ensemble average.

And this is a contrast that we will keep seeing, throughout, in this course, we will be dealing with ensemble averages and time averages; and very often, they will be equal, in for nice processes, the ensemble average and time average will be equal. Stochastic processes for which time average and ensemble average are equal are known as ergodic processes, and this leads to a whole theory called ergodic theory; but we will see some elementary building blocks of ergodicity in this chapter itself.

You can see that the answer is the same; it is already saying that there is something ergodic about the renewal process, the ensemble average and the time average. Well, here in the second statement, I did not have to say any almost surely or anything like that, why? $\frac{E[N(t)]}{t}$ will be a number for every t. So, as t tends to ∞, the limit will be a number; if at all it exists, it will be a number.

Whereas here, $\frac{N(t)}{t}$ is a random variable for every t. So, as t tends to ∞; I am looking at a limit of a sequence of random variables, so, I have to say in what sense it converges; I am saying it converges almost surely, in the strong law. Is it clear?

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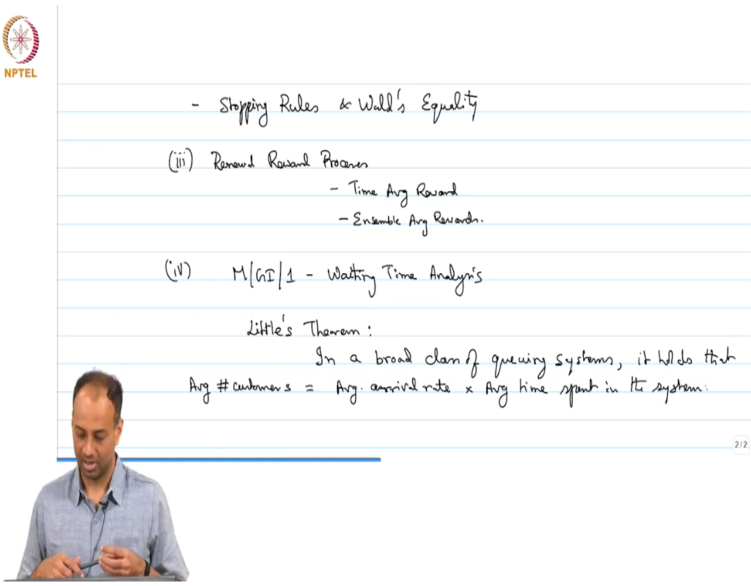

So, in proving the elementary renewal theorem, we will introduce something known as stopping rules, and we will prove an important relationship called Wald's equality, which should be very useful in its own right. We will study renewal reward processes. So, these renewal reward processes, you can think of as rewards associated with these renewal intervals. And so, we will again look at time average reward and ensemble average reward.

This renewal reward theory will also help us analyse some queueing systems. So, we will do $M/G/1$ waiting time analysis using renewal reward theory. We will prove a very important result in queueing known as Little's theorem. Little's theorem, let me just give you a preview of what it is. It basically says that, in a broad class of queueing systems, it holds that the average number of customers equals the average arrival rate times the average waiting time of each customer.

So, in plain English, Little's theorem says that, if you have almost any queuing system; so, I am not making this very precise; I have said broad class of queueing systems; certainly it is true in $G/G/1$; it is true even more broadly in some cases. It holds that the average number of customers in the system is equal to average arrival rate times the average time spent in the system.

Maybe I should write average time instead of waiting time, which means waiting in the queue; I should probably say average number of customers equal to average arrival rate times average time spent in the system. So, the average number of customers in a queueing system is equal to average arrival rate times average time spent by the customers in the queueing system. This is what Little's theorem is. It is very broadly applicable. We will state and prove this in a rigorous way, in this chapter.

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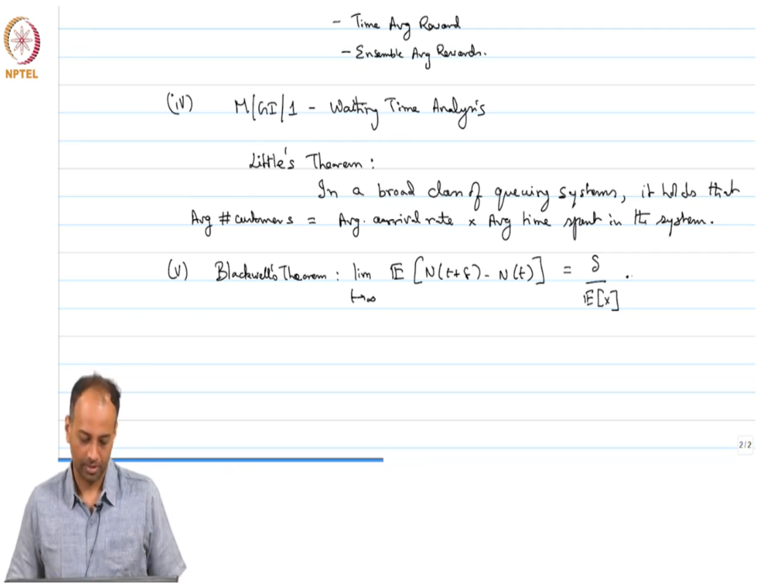

Finally, we will do Blackwell's theorem. You will do Blackwell's theorem most likely without proof. It looks at what is the expected number of renewals in a small; so, if you look at $N(t,\ t\ +\ \delta]$, you look at the $E[\ N(t, t\ +\ \delta]\ ]$ in a renewal process. What is this equal to, is the question. For the Poisson process, what is this equal to? Expected number of arrivals in; see, $N(t,\ t\ +\ \delta]$, right? This is what? $\widetilde{N}(t,\ t\ +\ \delta]$.

In a Poisson process, this is equal to $\lambda\delta$. So, in the renewal process also something like this is true. λ is what, here? $\frac{1}{E[X]}$. So,

$\lim\limits_{t \to \infty}\ E[N(t\ +\ \delta)\ -\ N(t)]\ =\ \frac{\delta}{E[X]}$ but it is not true for all t.

It is true in the $\lim\limits_{t \to \infty}$. And this is what Blackwell's theorem says. It is a very non-trivial result, but it is a very useful result.

This is true under certain more technical conditions; it is not unconditionally true always; but we will see this more rigorously later. So, this is the preview of what we are going to do in this chapter. Strong law is reasonably easy, in the sense that, once you understand strong law of large numbers for IID random variables, it is not difficult to understand why strong law holds for renewal processes.

Similarly, the time average reward theorem, renewal reward theorem for time average rewards which is in section 3, also follows along similar lines. The ensemble results are much more involved; the ensemble results, I mean elementary renewal theorem, the ensemble average rewards, they are, the answers are always the same, time averages and ensemble averages always meaning, almost always the same, but the route to proving them and the way we understand them this more subtle for ensemble averages.

Once you understand renewal reward theory well, we can understand $M/G/1$ waiting analysis very easily. And Little's theorem proof also comes partly using this reward theory. Blackwell's theorem proof is quite long and hard. I think I am inclined to skip the proof and just refer you to the appropriate reference, but we will; in this course, we are more interested in using it, as opposed to proving it.

Even the stopping rules or stopping times which we will discuss while proving the elementary renewal theorem, stopping rules are random variables which have certain properties. Ideally to define stopping rules properly, we need certain measure theoretic concepts. I will mention what they are, but we will stick to a definition which is easier to follow.

And rather than give you the measure theoretic definition, I think we can just stick to a simpler definition which will work for our purposes. So, throughout this course, we will always prefer clarity and applications over generality and rigour, because we are trying to apply it to queueing systems and other models. And we are not trying to prove everything very rigorously, but hope is that, when we are doing something that is not rigorous, I will tell you that; that is the approach we will follow. Good. So, this finishes the introduction of renewal theory.