Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology - Madras

Module - 3
Lecture - 21
Examples: Competing Poisson Processes

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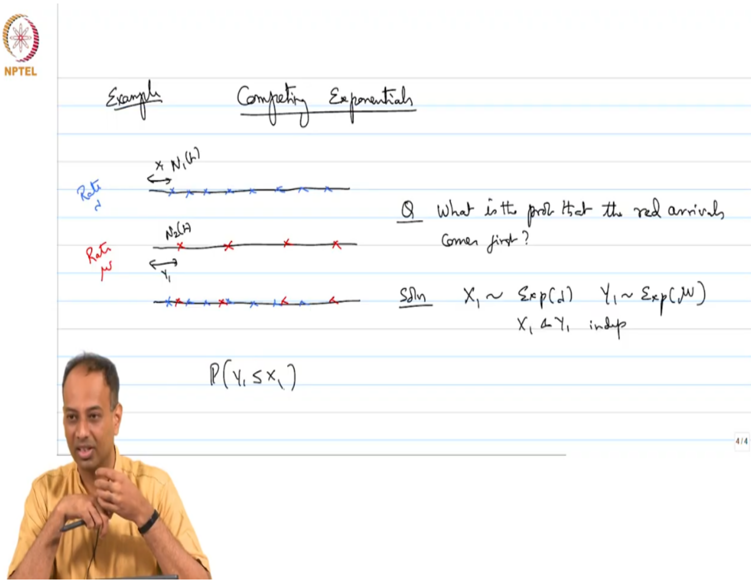

Let us do another example. Competing exponentials or competing Poisson processes. Let us say you have two Poisson processes, one is of rate λ and another is of rate μ. This is of rate λ; and independent. So, this guy is $N_1(t)$, this guy is $N_2(t)$. So, if you want to make it more colourful, you say that we are watching two radioactive samples emitting; this guy is one radioactive element, this guy is another radioactive element, they are throwing emissions at you; or you may be waiting for taxis or buses which are Poisson processes, does not matter.

So, you look at this combined process, we know which is a Poisson process of rate $\lambda + \mu$. So, you look at this combined guy. So, I have $N_1(t) + N_2(t)$; $N_1(t)$ goes here and all that. So, maybe I will draw a few points. So, I have all this. Now, I want to look at what is the; let us say you start at some time t; what is the probability that the red arrival comes first? So, I am starting at some time, which I can take by memorylessness to be the origin.

So, I am starting these two processes off; or you could start at some any time t and start looking at these two processes. Of course, it depends on; you could either have the red one coming first or the blue one coming first. What is the probability that the red arrival comes first? Red arrival means the μ guy, the μ process wins. So, that is why I have called it competing exponentials or competing Poisson process in this case; because, you have a, this red exponential, the red process is competing with the blue process, which means this μ exponential and λ exponential are competing.

And you are looking at the probability that the μ exponential wins. Because, starting at any time, the time for the red arrival is a μ exponential; the time for the blue arrival is a λ exponential; and they are independent, because they are independent processes. You want the probability that the red arrival wins. How do you calculate this? Solution: So, we can just take by memorylessness, we can just take that, we can just start time at 0.

You are just looking at; so, if you are looking at X_1, the time of the first arrival of the blue process, this is exponential with parameter λ; and Y_1 is exponential with parameter μ; and X_1 and Y_1 are independent. The red process wins if what? Is this notation? So, I hope you understand what I mean here, right? I am just starting at the origin, because it does not matter. They are all memoryless. I am calling this guy as X_1; calling that as Y_1.

So, I want the probability that the red process comes first, which means I am looking at the probability that Y_1 which is the μ process is less than or equal to X_1. Can I calculate this? How do I calculate this? **"Professor - student conversation starts"** So, did you say you will find the CDF of? Okay. Well, yeah, you can do that, you can find the; go ahead, calculate the CDF of $X_1 - Y_1$, and then integrate that out.

In this case, it so happens that; see, they are independent exponentials; you know the joint density, right? You can just integrate out the joint density. **"Professor - student conversation ends"**

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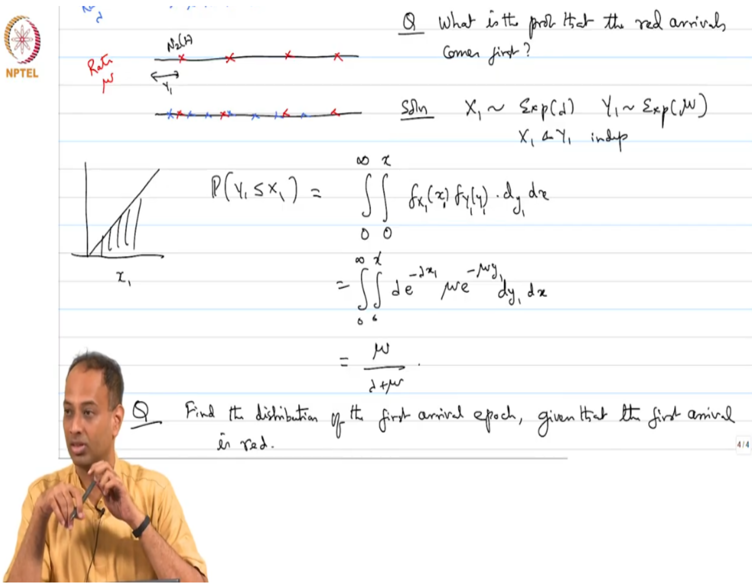

So, this will just be what? So, if I just have, that is my X_1, this is my Y_1; I want to look at the probability of that guy, right? So, in the joint density, I want the $P(X_1 \geq Y_1)$. So, this will work out to be what? This will work out to be; so, these are independent exponentials; so, I can write,

$$P(Y_1 \leq X_1) = \int_{x=0}^{\infty} \int_{y_1=0}^{x} f_{X_1}(x_1)\, f_{Y_1}(y_1)\, dy_1\, dx$$

So, maybe I should call this guy x_1; just, so that I am not integrating the same variable. I mean, I do not want to have the same variable here and here. $f_{X_1}(x_1)$, $f_{Y_1}(y_1)$. Can you work this out please? So, $f_{X_1}(x_1) = \lambda e^{-\lambda x_1}$, $f_{Y_1}(y_1) = \mu e^{-\mu y_1}$;so, this is just

$$P(Y_1 \leq X_1) = \int_{x=0}^{\infty} \int_{y_1=0}^{x} \lambda e^{-\lambda x_1}\, \mu e^{-\mu y_1}\, dy_1\, dx$$

So, this I can keep as x. I am just trying to not use the same variable inside and outside the integral; but you can work this out. What is this equal to? You should get $\frac{\mu}{\lambda+\mu}$. Just check if this is coming out to be correct. Next question; I will leave this as homework. So, let us say, find the distribution of the first arrival epoch, given that the first arrival is red. This is, you can calculate similarly.

I want you to find the distribution of the first arrival epoch, given that the first arrival was a red arrival. See, the distribution of the first red arrival is what? Exponential with parameter μ. But that is not what I want; I want the, given that the first arrival that came was red, what is the distribution of the first time, first arrival epoch? See, the distribution of the first arrival epoch is exponential with parameter $\lambda + \mu$, because you are competing exponentials.

You know this, right? You have, the minimum of two exponentials is an exponential with parameter $\lambda + \mu$. So, the first arrival epoch is exponentially distributed with parameter $\lambda + \mu$. The first red arrival epoch is distributed with parameter, exponentially distributed with parameter μ. That is not what I am asking. I am asking the distribution of the first arrival epoch, given that the first arrival was red.

You can work out a similar integral. What you will end up showing is that, it is still exponential with parameter $\lambda + \mu$. It does not matter whether the red one came first or blue one came first. You try working this out. It is somewhat surprising, because, if the blue process; let us say the blue process is much more intense than the red process, meaning that the first radioactive sample emits a lot of samples, and the red one does not emit very many samples.

You would think that if the first arrival was a red arrival, you waited a much longer time; it is not the case, that intuition is not correct. The probability of a red arrival coming first is small, but conditioned on the red arrival coming first, expected time still turns out to be, or the distribution of the time still turns out to be the unconditional distribution time. It is not at all trivial, you have to work it out. I mean, it is not a trivial meaning, you have to work out some double integral.