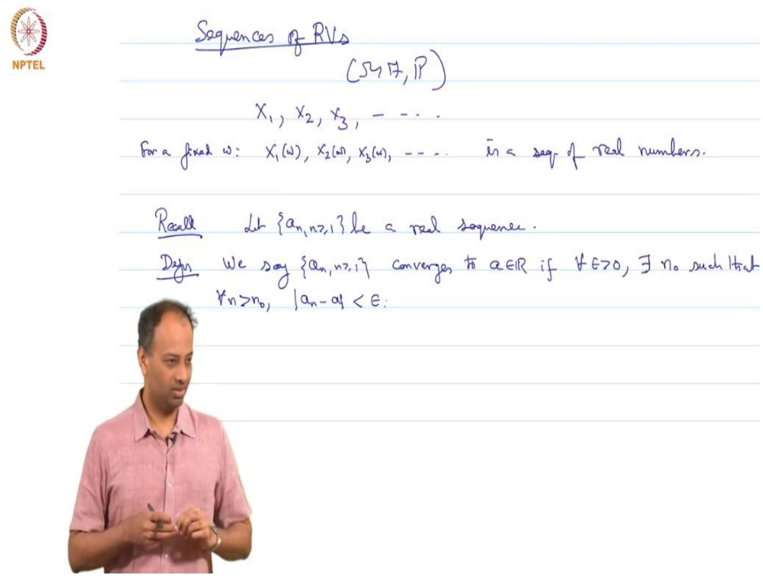
Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology - Madras

Module - 1
Lecture - 2
Sequence of Random Variables

(Refer Slide Time: 00:16)



Now, we will discuss Sequences of Random Variables. So, all of you know what a sequence of real numbers is, right? What is a sequence of real numbers? Little a 1, little a 2, ..., say it is a sequence of real numbers. What does that even mean? A sequence of real numbers is a mapping from natural numbers to real numbers. Now, so, sequence of real numbers for every natural number i, you get Ai as a real number.

So, indexing by natural numbers, you are talking about a sequence of real numbers. Similarly, a sequence of random variables is index by the natural numbers. And for each natural number i, you are talking about a random variable Xi, big Xi. That is what it is. So, again, please keep in mind that; so, Omega, F, P is always given, and all the random variables, the sequence of random variables come from this Omega, F, P.

They all come from the same sample space. So, once little omega realises, the entire sequence realises. So, you have a sequence of random variables, sorry; X 1, X 2, X 3, ... So, this is a sequence of random variables defined on the same probability space Omega, F, P.

And for any finite subcollection of the sequence, you can talk about its joint CDF. So, if I give you X 1, X 13, X 51, X 104, there will be some joint distribution, because there is an underlying P which induces a joint distribution on any finite subset of, any finite subcollection of these random variables.

And of course, when omega realises; so, for a fixed omega, you have X 1 of little omega, X 2 of little omega and so on. So, this is a sequence of real numbers. That is all. So, a sequence of random variables becomes simply a sequence of real numbers, once a little omega realises. Now, some other little omega prime realises, it will be some other sequence, X 1 of omega prime, X 2 of omega prime and so on.

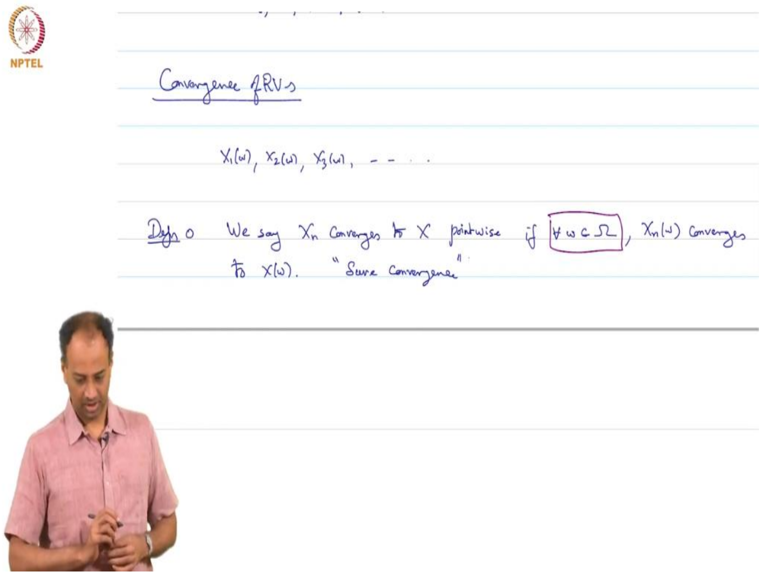
Now, when you talk about a sequence, the next thing you talk about is convergence. What does it mean to say a sequence converges? So, if I give you a sequence of real numbers a 1, a 2, ...; little a 1, little a 2, ...; I want to talk about convergence. What does convergence mean? Let us say, I want to talk about the sequence a, a n converging to some little a. So, intuitively, it means that, when n becomes large, the sequence a n gets very close, arbitrarily close to some, the proposed limit a, little a.

So, recall. So, let be a real sequence. Definition, we say, a n converges to a, if; when do you say n converges to a? If? **"Professor - student conversation starts"** No. That is what I want to define. So, if, for all epsilon, there exists n nought such that for all n > n nought, absolute value of a n minus a less than epsilon. This definition you already know. **"Professor - student conversation ends"**

So, you tell me epsilon, you give me whatever epsilon you want, 10 to the -6, 10 to the -8. I should be able to find a large enough n nought, such that, for all n > n nought, my sequence a n is within an epsilon band of a; it gets very close to a. How close is determined by this epsilon. And you can make epsilon as small as you like. So, and you can choose larger and larger n, for which a n gets closer and closer to a.

So, you know there exists for all notation, right; this is for all; this is there exists. I hope you know these notations already. We want to talk about convergence of random variables. This convergence of random variables, there are a few notions of convergence. I will briefly describe, let us say, 3 different notions, 3 are which are very commonly used.

(Refer Slide Time: 06:16)



Convergence of random variables; convergence of a sequence of random variables; that is what I mean. So, when little omega realises, so, real sequence realises. So, for a little omega, you can talk about whether that sequence X of, X 1 of little omega, X 2 of little omega, which is simply a real sequence, whether or not that has a limit. It is like talking about a limit of a sequence of real numbers, once little omega realises.

Now, you want to talk about these random variables converging. So, you talk about those omegas for which this sequence of real numbers, X 1 of omega, X 2 of omega converges or does not convert. So, just to; so, this omega realises and you get a real sequence ... Now, does this converge? And what does it converge to? Now, the problem is that, if at all it converges to anything, the limit will depend on omega.

So, for different values of little omega, you may not even converge, of course, right? The sequence may not even converge. But suppose it does converge to something, what it converges to will depend on little omega. So, for each little omega, you get a different limit, if at all the limit exists. So, the limit is also a function of omega. So, the limit will also be a; well, it is certainly a function of omega, whether it is a random variable is not obvious, because you have to have one more property.

See, the X 1, X 2, etcetera are random variables. But the limit, let us say X of omega, if at all it exists, is it a random variable? That requires one more technical condition which is measurability. Preimages of semi-infinite intervals have to be events. It turns out to be true. A limit of a; if at all a sequence of random variables has a limit, it will be a random variable. So, we can talk about the limit as also being a random variable with its own CDF and all that.

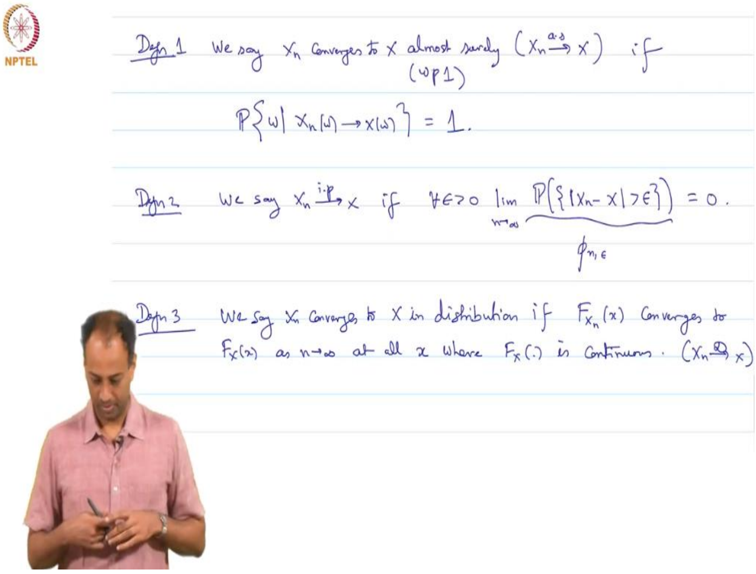
It is not obvious; it requires a proof, reasonably non-trivial proof. But you can, you can sort of take it; for this course, you can just take it on faith, okay? These things can be proven. If you can look at any graduate level text on probability, you will, you can prove that all liminf, limsup of random variables are all random variables. Okay, good. So, the limit is, if at all there is a limit, it must be a random variable, let us say X of omega.

And so, the limit we will talk was like X, just some random variable X. So, the definition 0. So, this is the zeroth definition of convergence. We say X n converges to X pointwise if for all omega in Omega, X n of omega converges to X of omega. So, you fix a little omega, X n of omega here becomes a sequence of real numbers. And you want; if that X n of omega converges to X of omega which is a real number, for every little omega, then you say X n converges to x.

The sequence of random variables converges to X pointwise. Pointwise meaning; so, pointwise means that. For every little omega in the sample space, you are demanding convergence to X of omega. So, X n of omega converging to X of omega, you already know, right, because it is a sequence of real numbers converging to another real number, which we have already defined. So, that you already understand.

Now, this guy is also known as, this pointwise convergence is also known as sure convergence, which is often too strong a notion of convergence to be useful in probability theory. So, the definition which is commonly used and the reason that I call the first definition the zeroth definition is because it is almost never encountered or used.

(Refer Slide Time: 11:39)



So, the definition which is more useful is what is known as convergence almost surely, meaning that you exempt some little omegas where X n of omega may not converge to X of omega. You exempt some little omegas, those which have probability 0. So, we say X n converges to X almost surely. Notation X n arrow X a dot s. Or, some people say, with probability 1. If the probabilities of those omegas for which is 1.

So, you are not demanding. So, X n of omegas converges to X of omega as in definition 0, but you are not demanding for all omega and big Omega. You are just demanding that those omegas where this convergence does take place has probability 1. There may be a few little omegas where this convergence does not take place. So, for those exempted little omegas the limit of X n of omega may not exist or may exist and may not equal X of omega; it may be something else.

But on a set of probability 1, set of full measures, so to speak, we have convergence. This is convergence almost surely. The little omegas that you are leaving out have 0 probability. That is all that it means. So, the set of little omegas where this convergence take place is a subset of the sample space which has full probability. So, there may be a few little omegas where the convergence does not take place, but this omegas where this convergence does not take place has 0 probability. That is all that it means.

So, there are two other commonly encountered definitions of convergence of random variables, which I will also put down. They are a little bit different from what we have seen so far. There is something known as convergence in probability. We say X n converges to X in probability. If for all epsilon > 0, limit n tending to infinity, probability that, equal to 0. So, this is the definition of convergence in probability.

So, here, you are not looking at; so, this is a little bit of a misnomer. So, here, it is not the sequence of random variables which is converging. Here, you are looking at simply a sequence of probabilities. So, you are looking at the difference between X n and X, the absolute value of the difference between X n an X; and you are taking some little epsilon. You are looking at the probability that the difference between X n and X exceeds this little epsilon.

So, you can call this some p of n, epsilon, if you like; it is a probability. This p of n, epsilon for every fixed epsilon is a sequence of probabilities in n. For each n, you get a probability. This sequence goes to 0. That is all we are saying. So, for large N, the probability that X n is different from X by more than epsilon, has very small probability. So, I mean, a better terminology for this would be to say that the sequence of probabilities converges, as opposed saying sequence of random variables converges, because this sequence of random variables may not converge.

It is just the sequence of probabilities that is converging. But terminology is that convergence in probability. Now, it turns out; it requires a proof, but it turns out that convergence almost surely always implies convergence in probability, but not the other way. Convergence in probability may not imply convergence almost surely. So, convergence almost surely is a stronger notion of convergence than convergence in probability.

Then, another notion of convergence is a convergence in distribution. We say X n converges to X in distribution if F X n of x converges to F X of x as n tends to infinity at all x where F X is continuous. So, this is denoted by X n arrow X with a D on top; convergence in distribution. Here, it is not the sequence of random variables which is converging, although that is what you say.

You say that X n is converging to X in distribution. It is really not the sequence of random variables which is converging, it is the sequence of CDFs. There is sequence of distributions functions that is converging. So, you take; each of these X n has a distribution function F X n of X. And you look at this F X n of X for a particular X. You send n to infinity. If that converges to the CDF of X, then you say, for every X, you are exempting certain X's where the F X is not continuous, if you are only demanding convergence of distribution at all the continuity points of F X.

So, this is a sequence of distribution functions that is converging. So, this is an even weaker notion of convergence. The reason is that you are only demanding that the distributions of the X n's gets close to the distribution of X. It is not that the values of X are getting close. Please realise. They may be very far away. So, in definitions 1 and 2, in some sense, X n's are getting close to X.

In the definition 2, for example, the probability of X n differing from X by much is very small. In definition 1, X n is actually converging to X, except maybe at a few points. But in definition 3, X n may not converge to X at all. They maybe have; even for very large n, X n of omega and X of omega may be very different. What is happening is that the distribution function of the X n's is converging. It is getting very close to the distribution of the X.

The values of X, X n's may not get close. So, this is an even weaker notion of convergence. So, what can be shown is that almost your convergence is stronger than convergence in probability which is stronger than convergence in distribution. So, convergence almost surely will imply convergence in probability and convergence in probability will imply convergence in distribution, but none of the opposite implications are through true.

These things can be shown. As it happens, there is a; so, in EE5110, these concepts are covered in much more detail. And as it happens, I have recorded that course on NPTEL before; some of you may be aware. Maybe I am plugging my own course, but there is a 2, 3 lectures on this topic. If at all you want to; I think it is a good idea to review those lectures, which are already on YouTube, if you want to know more about these concepts.