Stochastic Modeling and the Theory of Queues
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Module - 3
Lecture - 17
Conditional Arrival Density and Order Statistics - Part 2

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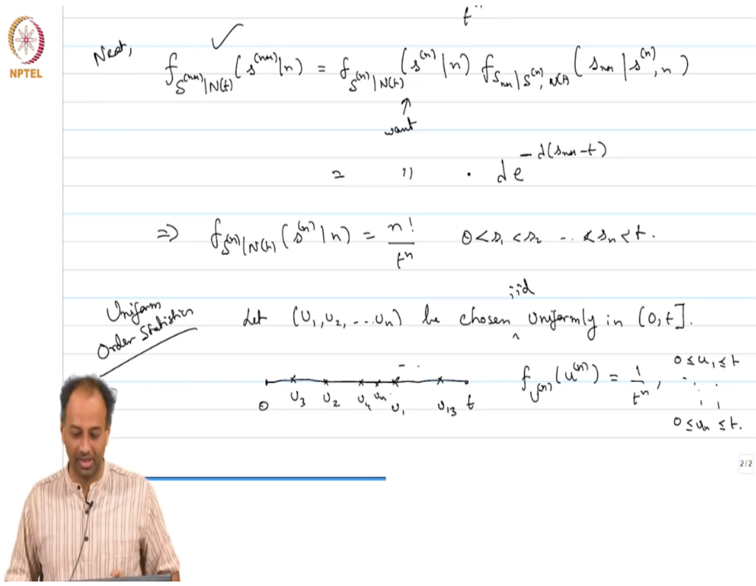

So, this guy is exactly like; so, you consider; so, this is the conditional joint density of the Poisson arrival epochs, given $N(t) = n$. Now, you consider uniform order statistics. Let $(U_1, U_2, ... U_n)$ be chosen IID uniformly in $(0, t]$. So, now I am saying, forget all about Poisson processes; you know what a uniform random variable is, right? So, if I give you some interval $(0, t]$, it is like uniformly throwing a dart on this interval.

You throw n darts independently. This is 0, that is t; I am throwing darts. First uniform, let us say may land here; let us say that is U_1. The second time, you throw another dart uniformly, independently; that may land here. Then you throw a third dart, it may land here; then the fourth dart may land here, and so on. And you throw basically n darts; they land wherever they land. The joint density of these $(U_1, U_2, ... U_n)$ is easy to calculate.

So, what is the density of any of these U_i's? $\frac{1}{t}$. So, what is the joint density of these $(U_1, U_2, ... U_n)$? The product, right? They are independent. So, the joint density should be $\frac{1}{t^n}$. That is clear, is it not? So, I can write

$$f_{U^n}(u^n) = \frac{1}{t^n}, 0 \leq u_1 \leq t, . . . 0 \leq u_n \leq t$$

See, what I am saying is that, if you take these; so, you are throwing these uniform darts; so, they land wherever they land; U_1 can land somewhere, U_2 can land somewhere. Now, you go ahead and sort these. Now, the U_3 in this case is the smallest, U_2 in this case is the second smallest, and so on. So, actually the largest does not have to be U_n, I do not know why I put U_n here. This could be, for all we can, this could be some U_{13}.

I do not know, right? U_n could be somewhere here. You see what I mean, they land wherever they land, but now you order them. Then you define an order statistics of these, meaning that you call the $min\ (U_1, U_2, ... U_n)$ as some variable, let us say Y_1. Then you say Y_2 is the second min of all this; Y_3, the third min of all this; so on till Y_n which is the max of all these guys. Now, you look at the joint density of not $(U_1, U_2, ... U_n)$, but $(Y_1, Y_2, ... Y_n)$. Now, what is that, is the question. It will turn out exactly like this, $\frac{n!}{t^n}$.

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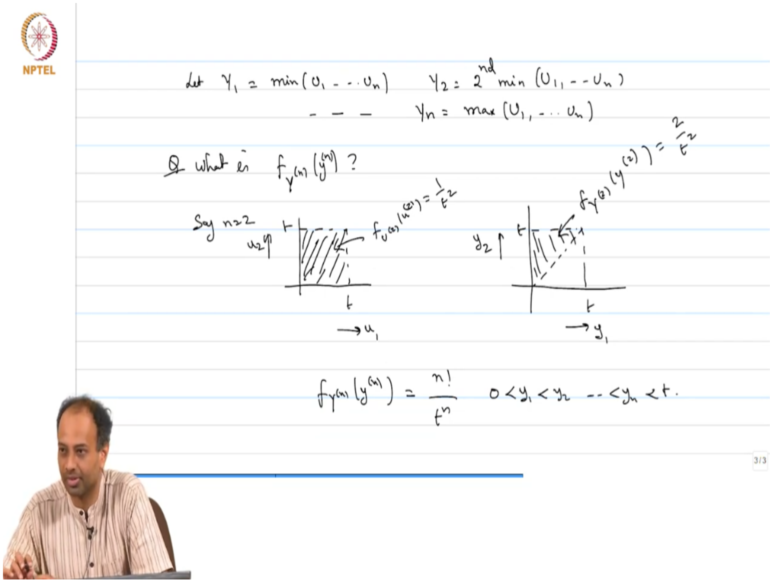

Let me just do that. So, now, order statistic uniform, what I mean is that, let $Y_1 = min\,(U_1, U_2, \ldots. U_n)$; $Y_2 = second\,min\,(U_1, U_2, \ldots. U_n)$; so, until $Y_n = max\,(U_1, U_2, \ldots. U_n)$. So, this $(U_1, U_2, \ldots. U_n)$ are uniform in $(0, t]$, and they can be in any order, right? The first one may land here, the second one may land here, third one may land here, and so on.

But these Y's are ordered; Y_1 is the smallest of the U's, Y_2 is the second smallest of the U's, and so on. So, these Y's cannot be anything; these Y's will have to satisfy $0 < y_1 < y_2 <. . < y_n < t$. Now, for this, what is the joint density, is the question. What is $f_{Y^n}(y^n)$? See, we can calculate this actually using a simple intuitive argument. Let us say there are only 2 variables, U_1 and U_2 which are uniform.

Let us say, so, for the U_1 and U_2; so, if you look at; this is $(0, t]$. I am looking at, this is the u_1; this is for u_2. I know that U_1 and U_2 are uniformly distributed in this square, independent uniformly distributed. So, the density will be what? The joint density of U_1 and U_2 will be $\frac{1}{t^2}$. And it will be; so, it is like coming out of the surface, it is constant, $\frac{1}{t^2}$.

Now, if you look at; so, you can see that in this region, the joint density is constant equal to $\frac{1}{t^2}$. $f_{U^{(2)}}(u^{(2)}) = \frac{1}{t^2}$. So, likewise, if you just look at the same $(0, t]$; but you look at, this is the axis for y_1, that is the axis for y_2. Now, what happens? Now, remember that $Y_1 = min\,(U_1, U_2)$, and $Y_2 = max\,(U_1, U_2)$. So, in that case, the density will be non-zero in which region? Only here.

So, if $Y_1 = U_1$ and $Y_2 = U_2$, you will have; that is one possibility. The other possibility is the other way, $Y_1 = U_2$ and $Y_2 = U_1$. Now, both of these have the same value of density, $\frac{1}{t^2}$. So, what essentially ends up happening is that this uniform density over the square, in the previous case, just folds over, so to speak, right? All this mass that lies below this line, just gets transferred here, because there are two possibilities of ordering.

So, here, $f_{Y^{(2)}}(y^{(2)}) = \frac{2}{t^2}$. And this is true for, in this region. This is for $0 < y_1 < y_2 < t$. So, this is exactly what is happening with n random variables, except now there are $n!$ possibilities. Let us say $(U_1, U_2, ... U_n)$ can be in any one of the $n!$ orders, and the joint density is of course constant, $\frac{1}{t^n}$. And corresponding to each one of these $n!$ orderings; see, there is only one $n!$ fraction of the n-dimensional hypercube, which will now be occupied, in the n dimensions.

Just like this, only half of the square is occupied. In n dimensions, there is $\frac{1}{n!}$ which will be occupied, and the density will have to be $n!$ times bigger. So, same sort of logic, except you can draw it. Same logic, there is nothing different. This 2 is 2!, if you like. So, what am I saying essentially? So, basically this argument can be formalised.

So, we are saying that the uniform order statistic has this as the density, for $0 < y_1 < y_2 <. . < y_n < t$, which is the same as the; so, this expression you agree, right? This is like $\frac{2}{t^2}$, except in the n dimensions, which is the same as the conditional arrival density of the Poisson point process. So, here is what I am saying. So, this result is saying something very powerful.

You take one scenario where a Poisson process is running and fixing some time t. It so happens that there are n arrivals. The distribution of the arrival times of these n arrivals will be statistically indistinguishable from just fixing this time and throwing n independent uniforms on this interval, and looking at the ordered distribution. So, if I give you two different; one coming from this Poisson process, as I said; in other case, I just throw n uniforms in $(0, t]$; you will not be able to tell the difference between what came from a Poisson process and what came from throwing uniform points on $(0, t]$.

What is the probability of two independent uniforms, say taking the same value? 0. They are continuous random variables; what is the probability of two continuous random variables which are independent taking the same value? 0. So, is this clear?

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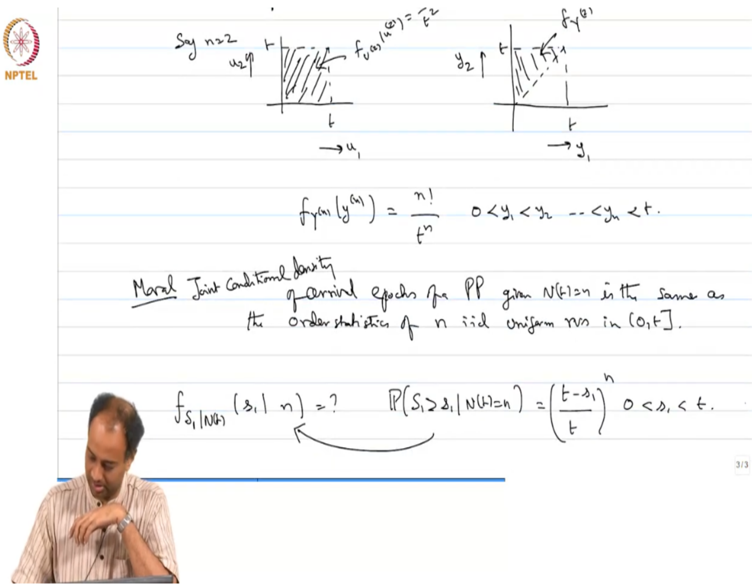

Now, this is the joint distribution. So, moral is that; Joint conditional arrival epochs of a Poisson process, given $N(t) = n$ is the same as the order statistics of n IID uniform random variables in $(0, t]$; they are statistically indistinguishable. So, whatever you know about n IID uniforms and these order statistics applies for the joint density of the arrival epochs, given $N(t) = n$; joint conditional density of arrival epochs.

Now, you can calculate things like this; $f_{S_1|N(t)}(s_1|n)$; equal to what? We will calculate this.

How can you calculate this? So, you can actually calculate the complementary CDF a little easier. This is like the smallest of the n uniforms. You know, for the smallest of n uniforms, you can calculate what the CDF is. So, for example, if you look at probability that; let me just write this down correctly.

$P(S_1 > s_1|N(t) = n) = \left(\frac{t-s_1}{t}\right)^n$. This is like all uniforms are bigger than this s_1. That is what this is. This is true for $0 < s_1 < t$. So, you differentiate this; you see what I mean, right? From this, you can; so, $P(S_1 \leq s_1)$ is 1 minus all that. Differentiate that to get this. From this, you can get this easily. So, this is an example.

Now that you have the joint density of $(S_1, S_2, \ldots S_n)$, given $N(t) = n$, you can also get the joint density of $(X_1, X_2, \ldots X_n)$, given $N(t) = n$. Would anything change really? The density will be the same for $(X_1, X_2, \ldots X_n)$, except that the constraints will now be; so, this constraint will have to apply, right? Now, this constraint will simply apply as $\sum_{i=1}^{n} x_i < t \text{ where } x_i > 0$.

So, you can also get;

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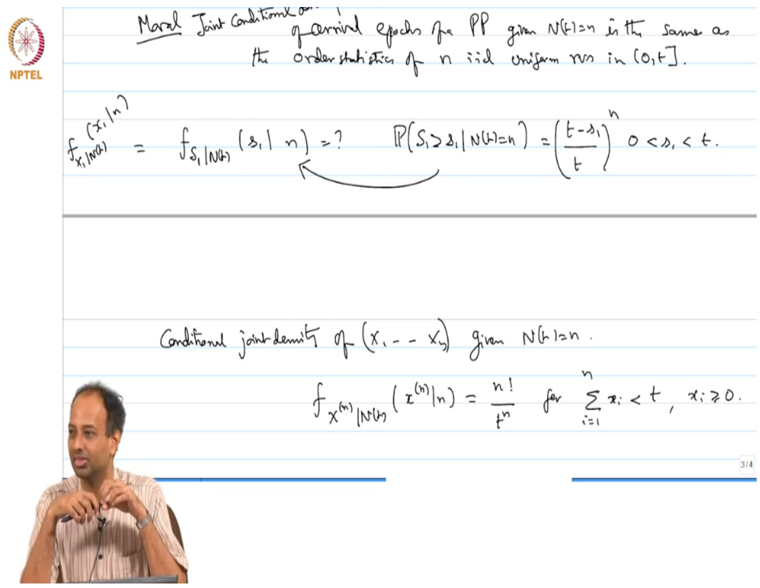

Conditional joint density of $(X_1, X_2, \ldots X_n)$, given $N(t) = n$. This will just turn out to be $f_{X^{(n)}|N(t)}(x^{(n)}|n) = \frac{n!}{t^n}$; the expression will be the same, because, after all, the density is constant; for, essentially you have this, $\sum_{i=1}^{n} x_i < t$ $where$ $x_i > 0$. So, it is the same density except that now $s_n < t$, and of course, each of these x_i's have to be non-zero.

So, this is symmetric. So, if you were to call X_{13} as X_9, and X_9 as X_{13}, then, it is completely symmetric. If you interchange the indices of two of these X_i's, their density would not change at all, which means that the density of each of these X_i's, the distribution of each of these X_i's, the conditional distribution is the same, which should not be too surprising.

In fact, we know the conditional distribution of S_1 which is the same as; note that this is the same as the conditional distribution of $f_{X_1|N(t)}(x_1|n)$, which will also be the same, which is equal to whatever, I can calculate from this CCDF. Once I know the marginal of X_1 the marginal of all the X_i's will have to be what? Exactly the same, because there is this perfect symmetry in this joint distribution. So, in the marginal densities you can easily calculate from here.

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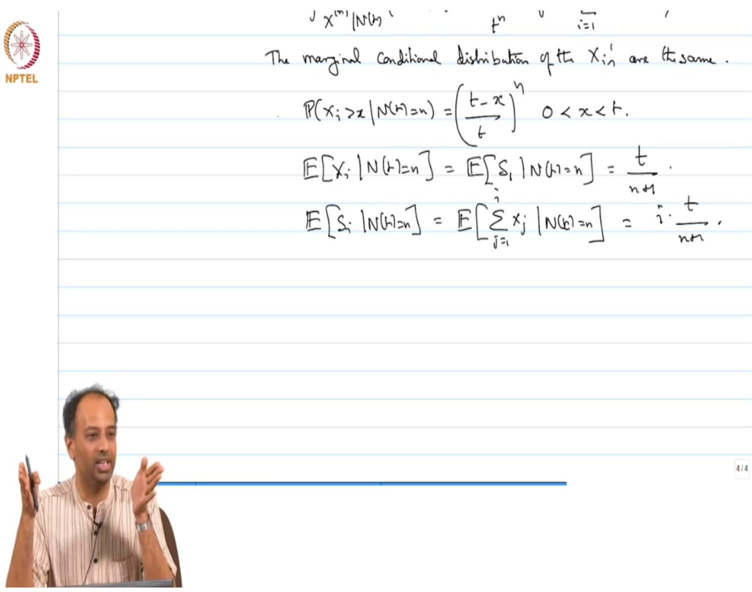

So, the marginal conditional distributions of the X_i's are the same $\forall\, i$. So, you can just write probability; so, you can get this;

$P(X_i > x \mid N(t) = n) = \left(\frac{t-x}{t}\right)^n;\ 0 < x < t$

Is that clear? See, I calculated the marginal density of S_1, the first arrival time, using the uniform order statistics property, which turns out to be the; S_1 is same as X_1; so, the density of X_1 is this; I mean, you can get from this.

So, but these X_i's are perfectly symmetric, so, each of these X_i's has this kind of a distribution. Is that clear? And also you can calculate; this is also something you can calculate. If you look at what is $E[X_i \mid N(t) = n]$, which is the same as $E[S_1 \mid N(t) = n]$or $E[X_1 \mid N(t) = n]$. So, S_1's density, this CCDF we know, right? If you calculate, what do you think you will get? Any intuitive guesses?

If you look at this picture; go back and look at this picture; where is that? Here. You are having; so, you know that you have n arrivals in $(0, t]$, right? And all these interval times, we said, have the same distribution. And so, how many intervals do you have here? 1, 2, n + 1. So, each of these intervals should be what? $E[X_i \mid N(t) = n] = \frac{t}{n+1}$; And also if you want this, what will this be? So, $E[S_i \mid N(t) = n] = \frac{it}{n+1}$.

So, they are all equal in expectation and distributed like so. Each of these X_i's, the conditional interval times are distributed like this, like what is given in this equation. Is that clear? So, now you can calculate all this very easily. I am just giving some examples of what you can calculate. So, the moral of the story is that, again, in an interval $(0, t]$, given that there are n Poisson arrivals, the arrival times of these n arrivals are exactly like some n uniform IID points thrown on the $(0, t]$. You cannot distinguish them statistically. So, we stop here.