Stochastic Modeling and the Theory of Queues
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Module - 3
Lecture - 16
Conditional Arrival Density and Order Statistics - Part 1

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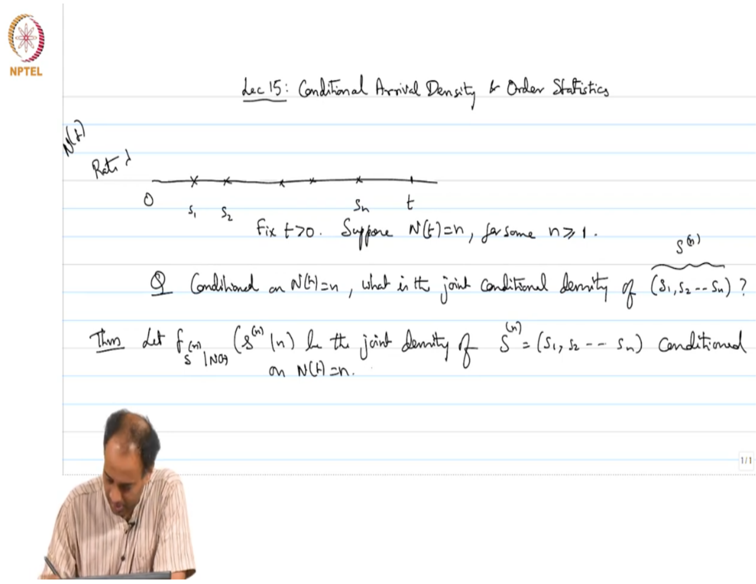

Good morning. Today we will discuss the conditional arrival density in a Poisson process. Let us say you have a Poisson process of rate λ. You fix a time. Suppose $N(t) = n; n \geq 1$. So, I am given that in $(0, t]$; I am fixing a time t and I am telling you that there have been n arrivals, and I now want to calculate the joint density of these arrival times, arrival epochs.

Of course, unconditionally, we know how the S_n's are distributed; the S_n will be an Erlang, the marginal being Erlang. The joint distribution also we know, we worked it out. We used induction to find the joint density of S_n unconditionally. Now, what I am saying is, fix your t, whatever you want; I am telling you that there have been n arrivals in it; conditioned on this, you tell me how they are distributed. That is the question.

So, question, condition on $N(t) = n$, what is the joint conditional density of the arrival epochs? This is the question. Is the question clear? Now, for a Poisson process, I will tell you what the main result is, and we will state it as a theorem. The main result is that if you fix this window $(0, t]$, and I am telling you that there have been n arrivals of this Poisson process; now conditioned on $N(t) = n$, in the $(0, t]$, these arrivals behave as though they are IID uniform in $(0, t]$.

So, if you see that there have been n arrivals of a Poisson process in $(0, t]$, it is statistically the same, it turns out as though somebody threw n uniform points on $(0, t]$, in an independent and; so, you take this $(0, t]$; you throw uniform points on this independently, n times; they will land somewhere. It turns out that the conditional statistics of these arrival times given $N(t) = n$ is the same as these IID uniform points thrown on $(0, t]$. That is the main result.

That is what you will state. Let f; so, this is like a vector, right? So, let me call that $S^{(n)}$; it is a vector consisting of $S_1, S_2 .... S_n$. Let $f_{S^{(n)}|N(t)}(s^{(n)}|n)$ be the joint density of $S^{(n)} = (S_1, S_2 .... S_n)$, conditioned on $N(t) = n$.

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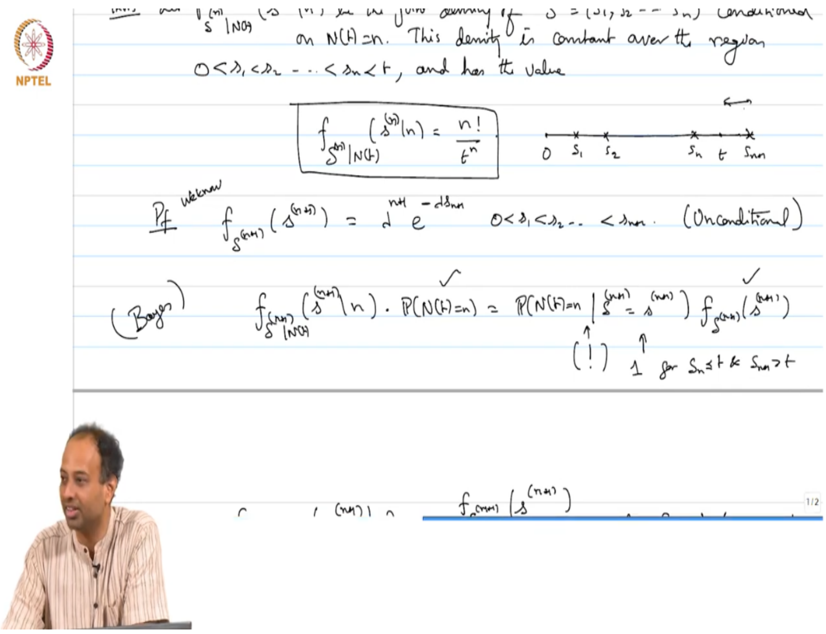

This density is constant over the region $0 < s_1 < s_2 <... < s_n < t$, and has the value $f_{S^{(n)}|N(t)}(s^{(n)}|n) = \frac{n!}{t^n}$; so, this is the joint density of $S_1, S_2 .... S_n$. On the right-hand side, you notice that there is no $s_1, s_2 .... s_n$, it is only a function of n and t. n is fixed, n is the number of arrivals; t is also fixed, t is the time interval. So, the function; this is supposed to be a function of $s_1, s_2 .... s_n$, but it is a constant function of $s_1, s_2 .... s_n$.

And the constraint set is this guy, $0 < s_1 < s_2 <... < s_n < t$. So, in that region, so, in n-dimensional space, this is some region, given by this constraint $0 < s_1 < s_2 <... < s_n < t$. So, in that region, this density is actually constant, this joint density is constant. So, I have to prove this first of all, and I also have to indicate that this is the same as the joint density of a bunch of uniforms, n uniforms lying in $(0, t]$, except they are ordered.

See, the $S_1, S_2 .... S_n$ are ordered. So, this is like the joint density of the order statistics of the uniform; you throw a bunch of uniforms independently; one of them will turn out to be the smallest, one of them will turn out to be the second smallest and so on. If you look at the joint density of the order statistics of independent uniforms, it turns out to be exactly this. That is what this result is saying. See, the proof is basically Bayes' theorem.

You have to work it out. Actually, your book gives two proofs. So, so, you know this, right? Let us look at the $n + 1$; so, this we know, $f_{S_{n+1}}(s_{n+1}) = \lambda^{(n+1)} e^{-\lambda s_{n+1}}$; so, I am just taking $n + 1$ arrivals. So, just to draw a picture; 0; so, this is $S_1, S_2 .... S_n$; t is somewhere here, and S_{n+1} is somewhere there.

But I am looking at; so, I am going to calculate first $f_{S^{(n+1)}|N(t)}(s^{(n+1)}|n)$, and from there, I will get $f_{S^{(n)}|N(t)}(s^{(n)}|n)$. So, in the picture, I have drawn, at t, they have been n arrivals exactly, in this picture. So, this comes from; have I written this down correctly? Can you please go back and check? I think this is correct, right? This is for $0 < s_1 < s_2 <... < s_{n+1}$; this is an unconditional density.

Now, what we can do is, we can just take $(f_{S^{(n+1)}|N(t)}(s^{(n+1)}|n))(P(N(t) = n)) = (P(N(t) = n|S^{(n+1)} = s^{n+1}))(f_{S^{(n+1)}}(s^{(n+1)}))$; this is by Bayes'. So, what have I done? See, I have just essentially written something like $P(A|B)P(B) = P(B|A)P(A)$, except that you have densities.

So, this is a density of S_n given n, and this is a conditional PMF, given the S_n's. Again, this is a little bit of a; I do not like this really, because, you are conditioning on continuous random variables taking certain values, which you really should not be doing; but the understanding is that, you are putting it in small intervals and taking δ to 0 and all that. With that understanding, so, maybe I should just put an exclamation mark here to warn you that this means something else.

So, this is the PMF of n, given these S_n's lie in small intervals and you send δ to 0; that is what this means. Now, if you look at this; so, I know this, right? This is the previous equation. And I know that of course. This is what this; this guy is the one I want. The first term is the one I want, isn't it? Almost; I want $f_{S^{(n)}|N(t)}(s^{(n)}|n)$, I have $f_{S^{(n+1)}|N(t)}(s^{(n+1)}|n)$. So, first let me get this.

So, this, the term that I marked with an exclamation is $(P(N(t) = n|S^{(n+1)} = s^{n+1}))$; whatever they are. Now, so, $P(N(t) = n) = 1$, if $S_n \leq t$ and $S_{n+1} > t$, like I have drawn in the picture; otherwise, it is 0. If S_n were here or S_{n+1} were here, it is 0. So, this guy is 1 for $S_n \leq t$ and $S_{n+1} > t$; otherwise it is 0. So, you restrict attention to the case when this holds. Then you will get what you want.

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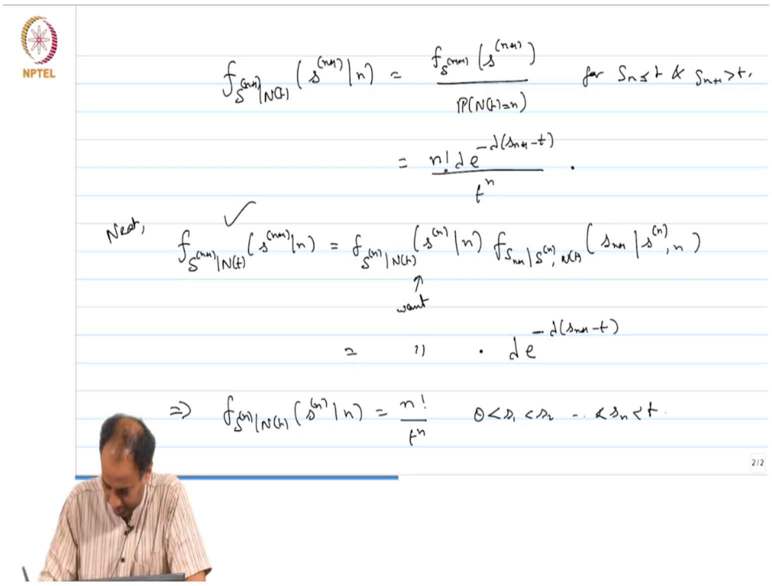

So, from this, you can get $f_{S^{(n+1)}|N(t)}(s^{(n+1)}|n) = \frac{f_{S^{(n+1)}}(s^{(n+1)})}{P(N(t)=n)}$; this holds when; $S_n \leq t$ and $S_{n+1} > t$. So, if this is the case, this term will be 1. So, I can write this conditional density equal to unconditional density over the Poisson PMF, this guy; this I already know. And this turns out to be whatever; we know the answer for this.

This is, if you just work this out, this will be $f_{S^{(n+1)}|N(t)}(s^{(n+1)}|n) = \frac{n!\,\lambda e^{-\lambda(s_{n+1}-t)}}{t^n}$.

So, this comes from this joint density, we already know, unconditional joint density; and $P(N(t) = n)$ is Poisson. Now, I got the joint density $f_{S^{(n+1)}|N(t)}(s^{(n+1)}|n)$. What I really want is the joint density of $f_{S^{(n)}|N(t)}(s^{(n)}|n)$. So, I can do some conditioning and find out what I want.

So, next, $f_{S^{(n+1)}|N(t)}(s^{(n+1)}|n) = (f_{S^{(n)}|N(t)}(s^{(n)}|n))\,(f_{S^{(n+1)}|S^{(n)},N(t)}(s^{(n+1)}|s^{(n)}, n))$.

So, this again follows from the definition of conditional density. So, what I have done is that; so, I have the joint density of $(n + 1)S_n$'s. Then I have written that in terms of the joint density of the first S_n's; and then, taken the S_{n+1} out separately; so, conditioned on S_n and $N(t)$.

This is just the definition of conditional density again. $N(t)$ is conditioned everywhere. So, assume this $N(t)$ is not there when you would agree with this straight away. There is just another level of conditioning. Now, so, what do we know in this equation? This we know. This is in fact what I calculated right here, the previous step. And this is the one I want, right? And this, do I know? This term, what happens?

So, you are looking at the conditional density of the $(n + 1)^{th}$ arrival, given all the first arrivals and $N(t) = n$. So, you are looking at this scenario where $N(t) = n$, and given all the previous interarrival times, what is the conditional density of S_{n+1}? So, what will that be? That will be another exponential. So, this interval will be exponential after all.

So, this will just be, $f_{S^{(n+1)}|N(t)}(s^{(n+1)}|n) = (f_{S^{(n)}|N(t)}(s^{(n)}|n))\ (\lambda e^{-\lambda(s_{n+1} - t)})$.

So, if you look at it, this left-hand side already has that $(\lambda e^{-\lambda(s_{n+1} - t)})$. So, all I am saying is that, this gets bumped out of the expression in what I want. So, this just implies, $f_{S^{(n)}|N(t)}(s^{(n)}|n) = \frac{n!}{t^n}$. And this is true for $0 < s_1 < s_2 <... < s_n < t$; this is what you want. Any questions on this?

It is mostly a, I mean, it is just Bayes' rule and a sort of mechanical derivation, and one point we have invoked memorylessness, this S_{n+1}'s arrival, we have invoked this memorylessness. But the form of this conditional joint density is very interesting, as I mentioned earlier.