Stochastic Modeling and the Theory of Queues
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Module - 2
Lecture - 15
Example: Poisson Splitting

Shall we do a couple of examples, just so that you know how to apply these things.

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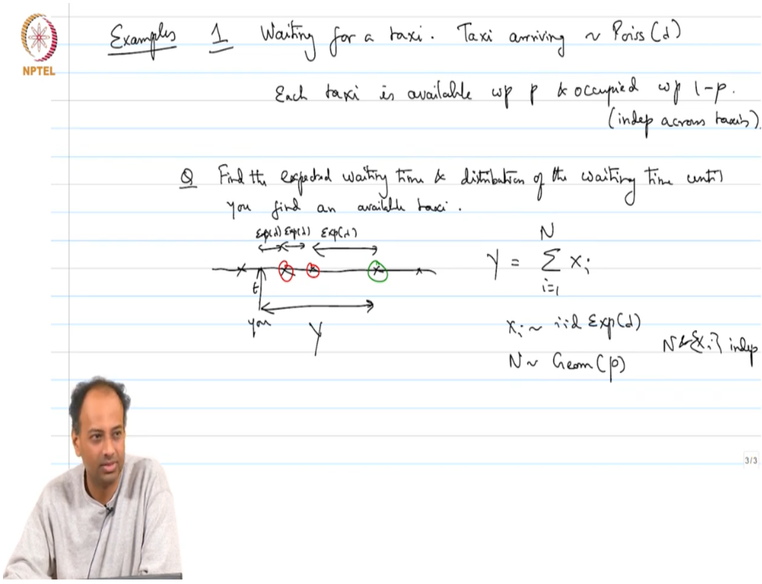

Examples: So, you are waiting for a taxi. So, you are standing somewhere; you are waiting for a taxi; and taxis are arriving according to a Poisson process of rate λ. Each taxi is available with probability p and occupied with probability $1 - p$, and this is independent of other taxis. Now, I am interested in; what am I interested in? I am waiting for a taxi. They are coming according to a Poisson process.

So, the time for; so, I show up at some point; the Poisson process is running anyway. So, the amount of time I would wait until I see a taxi is; what? Is an exponentially distributed random variable with parameter λ, except there is a catch. The taxi that comes first may already be occupied. I want a taxi which is not occupied. So, the question is, how long do you wait to actually find a taxi which is available?

So, the question is, find the expected waiting time and distribution of the waiting time until you find an available taxi. Is the question clear? Now, see, in this problem, see, you can imagine that there is some Poisson process which has been running anyway. So, you can say that there is some Poisson process that is running, which is taxis. And you say, you show up at some time; this is you, you suddenly show up at some time t.

And then, you start looking for a taxi. The first taxi shows up here. It may be occupied. Second taxi may be occupied; the third taxi may be available. So, then you end up waiting. So, let me say this is not available; this is not available; and that guy is available. I want to find the expected time until you find an available taxi and the distribution of this time. Now, as it happens; see, this is a Poisson process.

So, this time t, for all practical purposes, can be regarded as 0; because, you fix a time t and look ahead, what you see is another Poisson process, because we established that the time until first arrival is an exponential, which is independent of everything in the past and all that. So, for all practical purposes, this t can be taken as the origin. So, these are all; see, all these inter-arrival level times; so, this guy, this is time t.

So, now, this guy, this interval, this interval, they are all IID exponential λ. Why? The process is a Poisson process. So, I wait for a certain number of exponential intervals. So, each of these intervals is exponentially distributed with parameter λ, and I wait for a certain number of exponential intervals. How many exponential intervals do I wait? It is also a random variable.

So, the number of taxis that I have to see before I find an empty taxi, an unoccupied taxi, is also a random variable. What kind of a random variable is it? It is a geometric random variable. You see why? See, we think of these taxis as being; taxi being occupied or not occupied is a success-failure trial, it is a Bernoulli trial. So, success probability is p; failure probability is $1 - p$; and you are waiting for first success.

So, the number of trials before your first success is geometric. So, if I take this as the time Z, this Z is nothing but the sum of X_i's, where X_i's are the inter-arrival times. See, I used Z for the first arrival; maybe I should use some other variable; let me call this Y, because I used Z for the first arrival, that theorem; maybe I should call this Y.

$Y = \sum_{i=1}^{N} X_i$. Now, what is this X_i? X_i's are IID exponential with parameter λ.

And N is; see, the N is the number of taxis I have to see until I find an unoccupied taxi. This is like a number of trials I have to go through, Bernoulli trials I have to go through, until I see my first success. So, N is geometric with parameter p. You know what a geometric random variable PMF is, right? So, I am interested in calculating the distribution of a geometric sum of IID exponentials.

Now, you might have seen this in your undergraduate probability course as to how to calculate the distribution of random numbers of these IID random variables. Now, by the way, N and X_i are independent. Why? So, whether a taxi is occupied or not occupied is independent of the arrival times. The Poisson process is independent of these occupied, non-occupied random variables. So, N and X_i are independent.

So, this is a sum of a random number of exponential random variables, where the number N is independent of each of the X_i's also. So, in undergraduate probability, you might have calculated this; I do not know; do any of you remember how to do this? You can condition on $N = n$ and proceed.

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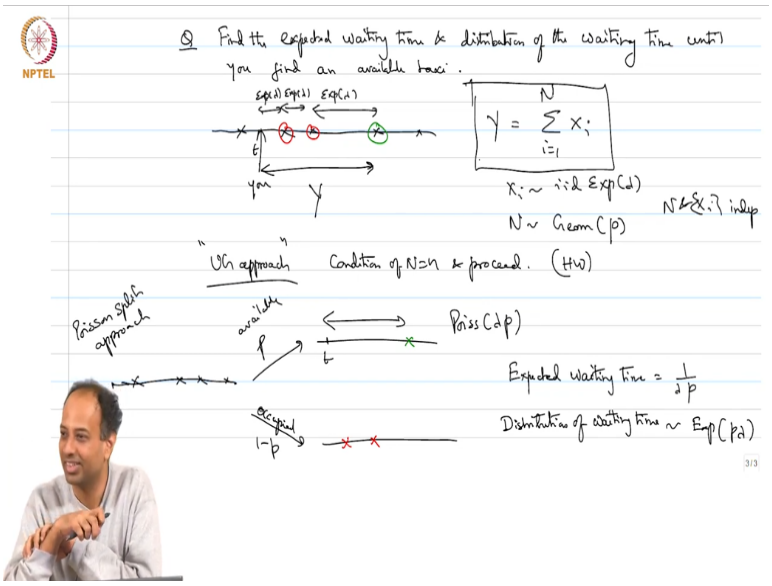

So, what may be called the UG approach; not to be insulting to undergraduates, right? Is to condition on N and proceed. So, you will apply the law of total probability. What I mean is, condition on $N = n$, you will have, the Y will be Erlang with parameter n^{th} order Erlang, because you see n. And then, n itself is geometrically distributed; you can fight it out; apply the law of total probability and fight it out. I suggest you do this.

So, this, what I condescendingly call the undergraduate approach, I will leave it to you as homework. But the whole point of this exercise is to; if you know your Poisson process, you can get it in one stroke. Namely that; see, this occupied and unoccupied is another way of saying, I am splitting the process. So, you can view this splitting as; so, you send it up when the p is the probability of it being available, isn't it?

Available is probability p; so, you send it up with probability p; send it down with $1 - p$. You are waiting for the first arrival in the up process. But the up process is what kind of a process? The Poisson process of rate λp. So, your time for the first taxi to arrive, which is actually available, which is the first arrival in the top process, is what? Expectation of the time will be $\frac{1}{\lambda p}$, and the distribution will be exponential with parameter λp.

Finished; there is nothing more to it. So, if you did not follow what I spoke out; so, this is your original process of taxis. So, if it is occupied, you send it down; if it is available, you send it up. See, all the red arrivals will go down. So, in my picture, the first two were not available, so they went down. And the third one was available, so that went up. So, this means available; this means occupied.

So, imagine, the taxi process is being split, and of course, each taxi is independent of the other taxis and it is independent of the interarrival times; all that is assumed. So, essentially, you are waiting for the first arrival in; so, the t was here I think, in my picture; it does not matter where t is, right? It could just as well be the origin, as I argued. So, let us say, this is my; t is actually, let us say here.

So, I am waiting for some time t till my first arrival in that process. But this is a Poisson process of rate λp. So, the expected time I wait; is what? $\frac{1}{\lambda p}$. And the distribution of waiting time is; so, this waiting, will be exponential $p\lambda$. So, this is the Poisson splitting approach. So, if you know Poisson processes and the splitting and all that, you can get this in one stroke.

So, what have we actually shown, if you think about it? What have we shown? We have that this guy, which is a geometric sum of IID exponentials; you have shown that this guy is also an exponential. Why? What you have accomplished by this argument is that, if you sum a geometric number of IID exponential random variables with N being independent of X_i, Y that results as the sum, is also an exponential with parameter $p\lambda$, where λ is the parameter of these exponentials and p is the parameter of the geometric.

We have proved that in a very elegant way. Of course, you can prove it using this conditioning on $N = n$; write out the Erlang; you can do all that, you will still get the same answer. You can do it and see.