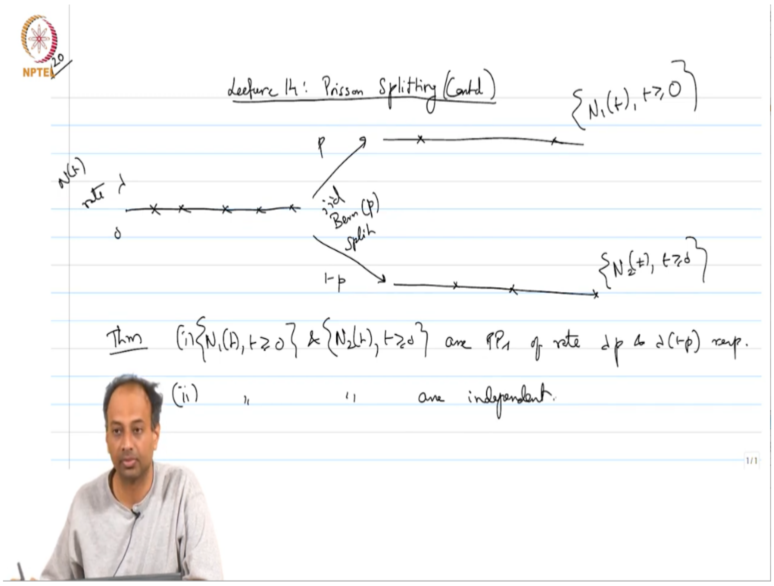
Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology - Madras

Module - 2
Lecture - 14
Splitting of Poisson Process - Part 2

(Refer Slide Time: 00:16)



Yesterday we started discussing the splitting of the Poisson process. The setting is that we have a Poisson process of rate λ running, and you toss a coin every time there is an arrival, independently each time. And if your coin shows head, you send the arrival up; if your coin shows tail, you send the arrival down. This results in two counting processes $\{N_1(t),\ t \geq 0\}$ and $\{N_2(t),\ t \geq 0\}$. What we said is that, for these IID Bernoulli; this is called IID Bernoulli splitting, Bernoulli p split.

We said that $\{N_1(t),\ t \geq 0\}$ and $\{N_2(t),\ t \geq 0\}$ are Poisson processes of rate λp and $\lambda(1 - p)$ respectively. And these two processes are independent. So, it is not surprising that the up process is of rate λp, because you are sending a fraction p of the arrivals up. And likewise, down process is of $\lambda(1 - p)$. What is surprising is that they are both Poisson processes.

And even more remarkably that the two processes that result from the same process, splitting the same process are in fact independent. That is a very remarkable property. Yesterday, we showed this $o(\delta)$ approach, to prove that the split processes have the correct distribution. There you got, the process had a Poisson IID Bernoulli sort of increments with parameter $p\lambda$.

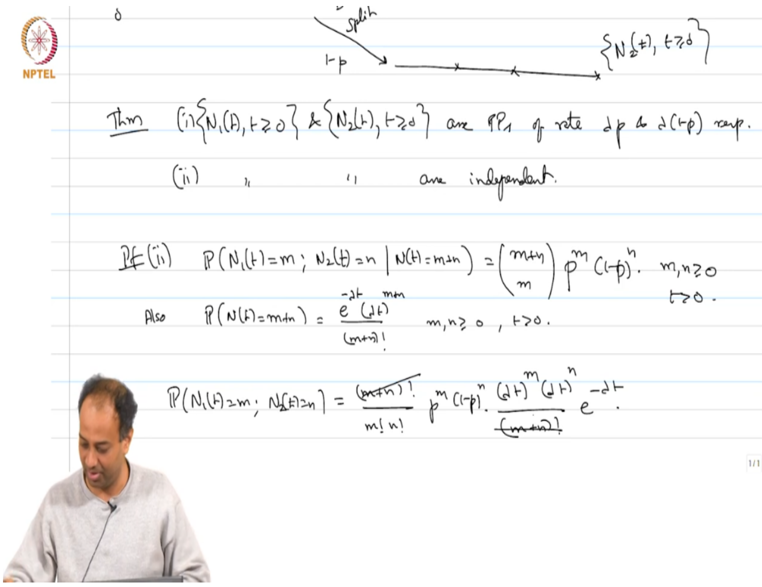
And SIP and IIP would follow from SIP and IIP for the original process, and the fact that these Bernoulli splits are independent. So, this much we showed yesterday, we wrote it out and showed yesterday. What we cannot show using this $\lambda\delta$ approach is that these two processes are independent; that you cannot show. So, the approach we took in the last lecture, this $o(\delta)$ term makes all the difference.

This $o(\delta)$ is essentially what induces this independence, as I argued yesterday. Because, if, suppose, in the original process, you had one arrival with probability $\lambda\delta$ and zero arrivals with probability $1 - \lambda\delta$, no $o(\delta)$ involved. So, the probability of having 2 or more arrivals is precisely 0. If this were the case, then you can show that if an arrival went up, in that micro slot you cannot have an arrival in the down process, which means that these two processes cannot be independent.

Because, if I know something, if I know there is an arrival in the first process, there is no arrival in the second process, and they are not independent. Now, because the probability of 2 or more arrivals at the same time is not exactly 0, it is close to 0, but it is not exactly 0, so, there is ever so small a probability that 2 arrivals can come at the same, in the same micro slot. This independence comes via that, in some sense.

So, but this $\lambda\delta$ approach is not fine enough to give us this independence; you cannot prove it using this $\lambda\delta$ or $o(\delta)$ approach. So, for this, we have to directly look at the; so, we have to take a direct approach. Basically, we will use the Poisson PMF of $N(t)$. Now, the way we do this;

(Refer Slide Time: 05:09)



So, proof of 2. I indicated how 1 is proved. So, the $\lambda\delta$ approach is okay to prove 1; not enough to prove 2. Proof of 2: You look at something like this. You look at the $P(N_1(t) = m, N_2(t) = n \,|\, N(t) = m + n)$; you fix a t first. You look at this conditional probability. You fix a t and you look at the conditional probability that, given there has been $m + n$ arrivals in the original process, you are looking at the probability that m were sent up and n were sent down.

What is this equal to? So, you had $m + n$, you are given that you have had $m + n$ arrivals in the original process, you are essentially looking at the probability that, of these $m + n$; every time there is an arrival, you toss a coin, and of these $m + n$ coin tosses, you want exactly m heads and n tails. What is the probability of that? This is a binomial. So, this will be,

$P(N_1(t) = m, N_2(t) = n \,|\, N(t) = m + n) = \binom{m+n}{m} p^m (1-p)^n$; this is true $\forall m, n \geq 0, t > 0$

Now, we also know; see, the $N(t)$ is a Poisson process. So, what is the $P(N(t) = m + n)$? $P(N(t) = m + n) = \frac{e^{-\lambda t}(\lambda t)^{m+n}}{(m+n)!}$, because $N(t)$ is a Poisson process; I know the Poisson PMF. So, I can write the joint distribution as; so, if I look at this guy,

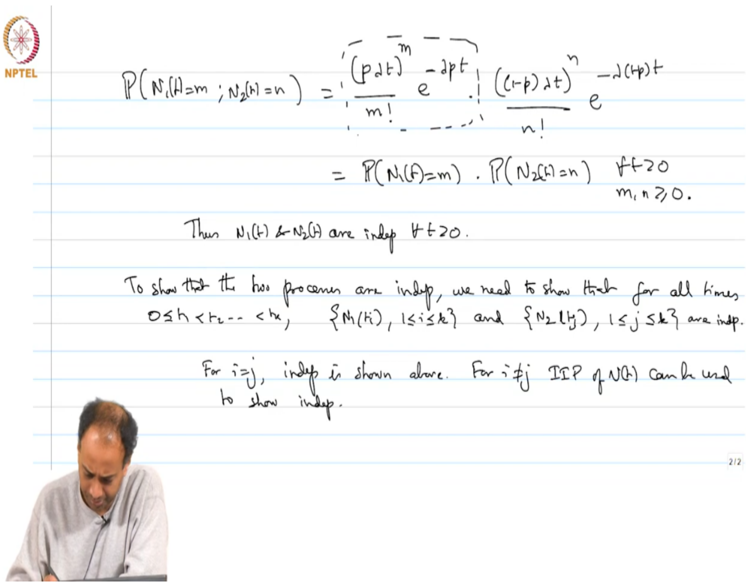
$P(N_1(t) = m, N_2(t) = n)$, this is just the product of these two, by definition of conditional probability. So, I will get what?

Let me just write, $P(N_1(t) = m, N_2(t) = n) = \frac{(m+n)!}{m!n!} p^m (1 - p)^n \frac{(\lambda t)^m (\lambda t)^n}{(m+n)!} e^{-\lambda t}$

What have I done? I have just written out this binomial term as, in terms of these factorials. And this $(\lambda t)^{m+n}$, I split it like this. I think I have missed a term. So, there should be $e^{-\lambda t}$. Thank you.

So, this guy goes off; good thing. And then, I can; basically, this $(p\lambda t)^m$; now, I can; basically, I can marry this and this, I can marry this and that, they are all power m power n.

(Refer Slide Time: 09:05)



So, let me write this again. $P(N_1(t) = m, N_2(t) = n) = \frac{(p\lambda t)^m}{m!} e^{-\lambda pt} \frac{((1-p)\lambda t)^n}{n!} e^{-\lambda(1-p)t}$

This is a Poisson PMF with parameter $p\lambda$.

And likewise, this is the Poisson PMF with parameter $\lambda(1 - p)$. See, in the previous proof, I have already shown that $N_1(t)$ is a Poisson process of rate $p\lambda$, using the $\lambda\delta$ technique. So, this Poisson PMF simply represents $P(N_1(t) = m)$. And likewise, this represents $P(N_2(t) = n)$. This is true for all $t > 0$ and any m, n non-negative integers. So, what have we shown?

I have just shown that the joint PMF of $N_1(t)$ and $N_2(t)$ factorises. What does that mean? What and what are independent? $N_1(t)$ and $N_2(t)$ are independent for all t. Thus, the random variables $N_1(t)$ and $N_2(t)$ are independent for all t. So, am I done proving independence of these two processes? I have proved it for all t. What have I proved? I have finished proving that for all t, the random variables $N_1(t)$ and $N_2(t)$ are independent.

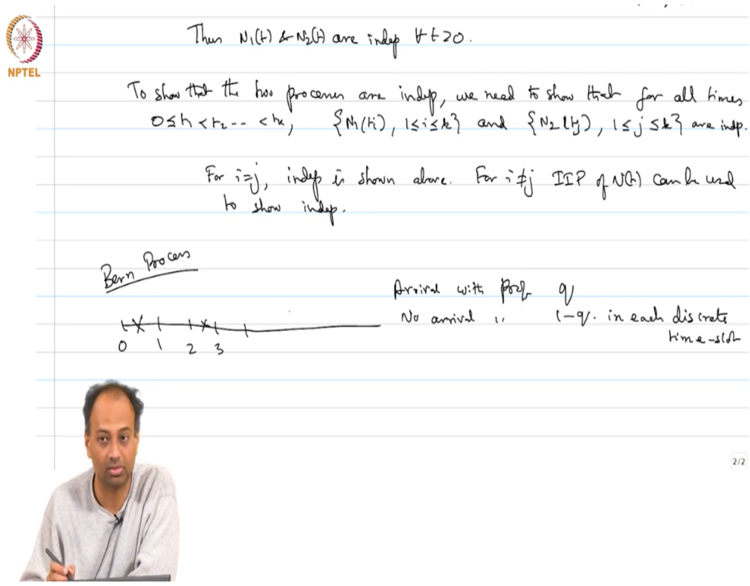
Is that enough to show that the two processes are independent? This is true for every t, but it is not enough to show that the two processes are independent. To show that the two processes are independent, you have to use, we need to show that for all times $0 \leq t_1 < t_2 <... < t_k$. You want to show that $\{N_1(t_i), 1 \leq i \leq k\}$ and $\{N_2(t_j), 1 \leq j \leq k\}$ are independent. See, because, just showing that $N_1(t)$ and $N_2(t)$ are independent random variables for every t is not enough; you have to prove that every finite collection of the $N_1(t)$'s, $N_1(t_1)$, $N_1(t_2)$, et cetera is independent of any other finite collection of, from the $N_2(.)$s.

See, what we have shown is that, if $i = j$ in these 2 indices, then I have shown that $N_1(t_i)$ and $N_2(t_i)$ are independent. So, for $i = j$, I can use whatever I have proved so far. So, for $i = j$ independence is shown above. Now, for $i \neq j$, what do you have to use? If I look at $N_1(t_1)$ and let us say $N_2(t_2)$ or some other collection, what do I have to show? I have to show that they are independent. Now, the original process has IIP.

So, since the original process has IIP, I can use IIP of the original process to show IIP implies independence; IIP of $N(t)$ can be used to show independence, for $i \neq j$. You can write it out and see. It is not a difficult thing to do. So, that proves part 2 of the statement, which is that these two Poisson processes that come from the split are in fact independent Poisson processes.

So, in some sense, this Poisson process is very much like a Bernoulli process in almost all respects except that, when you split a Bernoulli process, you do not get two independent processes.

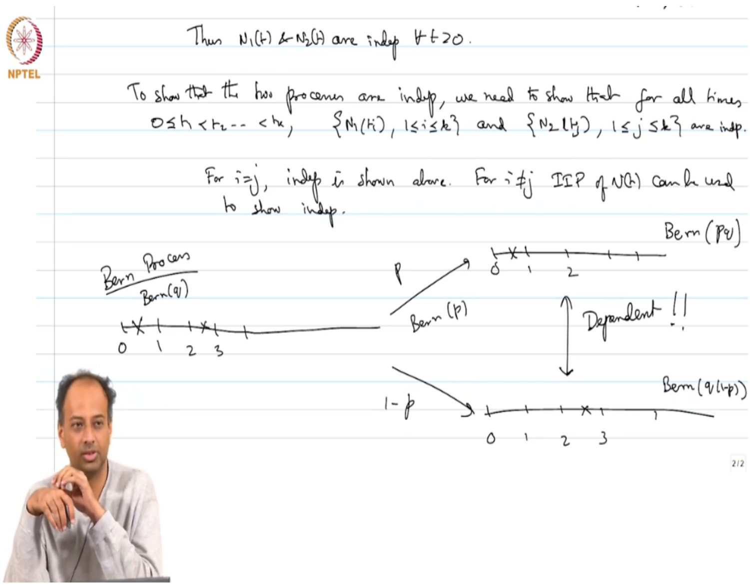
(Refer Slide Time: 15:36)



So, if you just take this Bernoulli example; you know what a Bernoulli process is, right? So, for the Bernoulli process, time is split into discrete intervals. And in each interval, you have ; so, this is like 0, 1, 2, 3; it does not matter; it could be δ_2, δ_1; I do not care. So, Bernoulli process has an arrival with probability q; no arrival with probability $1 - q$, in each discrete time slot, and these are independent across time slots.

So, here, you have an arrival with; this is discrete time slots, this is not continuous time. So, you have an arrival with probability q here; there may be no arrival here; there may be an arrival, and so on. Now, you split this guy with another Bernoulli coin toss, which is independent of the process. So, maybe this is a bad place to write all this.

(Refer Slide Time: 17:10)



So, I am going to write the split here. I am going to split; this process is Bernoulli q; this process is Bernoulli p. So, this again is a discrete time process, 0, 1, 2 and so on. So, I am taking a Bernoulli process which is Bernoulli q and splitting with another Bernoulli process which is Bernoulli p; and the splitting process and the arrival process, the original Bernoulli q process are independent.

So, here, what will happen is that; you can show that this is a process which is Bernoulli of what? pq. Because you have an arrival here, if and only if you had an arrival in the original process and your coin toss sent it up. And likewise, this down process will be Bernoulli q times $1 - p$. The down process will be a Bernoulli process with arrival probability $q(1 - p)$, any time slot, same logic.

But what is the big difference? These two processes are dependent; because, if I do see an arrival here, there cannot be an arrival here. So, given that this process conditioned on there being an arrival here, you know for a fact that there cannot be any arrival here. Whereas, if you split a Poisson process, you get two Poisson processes, which is nice. But what is even more remarkable is, you get independent Poisson processes.

And this independence, as you see, here there is just q or $1 - q$. There is no $o(\delta)$ involved. And because of that $o(\delta)$, which is that innocuous thing, which is mostly a nuisance, is actually giving you this independence. So, let us not be, let us show some respect to this $o(\delta)$

. So, it is very important; it is a crucial aspect of the Poisson process. So, that finishes the module on splitting.