Stochastic Modeling and the Theory of Queues
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Module - 2
Lecture - 13
Splitting of Poisson Process - Part 1

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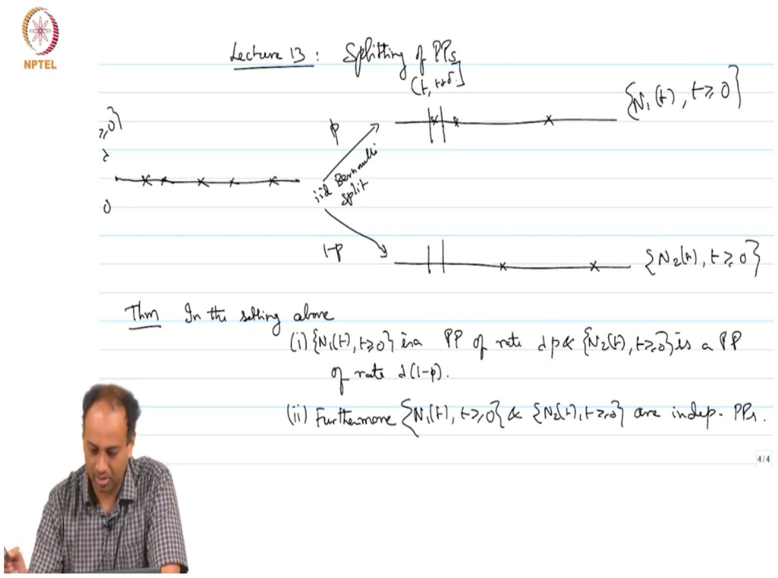

Now, we can look at splitting. So, this is in some sense the opposite of what we did. So, there you took two independent Poisson processes and merged them. Now, what you are going to do is take one Poisson process and split. How are you going to split? By tossing a coin. So, here is a Poisson process of rate λ. So, this $\{N(t),\ t \geq 0\}$ is of Poisson process of rate λ. Now, I have a coin, a p coin with me; a p coin, meaning it shows up heads with probability p and shows up tails with probability 1 - p.

And this coin is tossed independently, every time I see an arrival in the process. And this coin has nothing to do with the process itself; the coin is independent. The coin toss is done every time there is an arrival. So, imagine you are looking at some radioactive decay. Every time there is an emission, a counter clicks, you toss a coin. If it shows up heads, let us say you send it up; send it up meaning, you consider, you count it as an up process, so to speak.

And if it shows up tails, which is probably 1 - p, you send it down, you consider it as a down arrival. So, what happens? So, the first arrival showed up, you toss a coin, and this coin toss is independent of these exponential random variables that are coming at you. So, the first coin is, let us say the first arrival went up. Next, the second arrival comes, you toss the coin. Now, this coin tosses are independent of your previous tosses and any other inter-arrival times, et cetera you have seen from the process.

It is an independent coin toss. Let us say, that also went up. The third one may go down; the fourth one may go up; the fifth one may go down and so on. Now, this is called IID Bernoulli splitting. This is the IID Bernoulli splitting. You take a Poisson process of rate λ, and every time there is an arrival, you toss a coin. If it is heads, you send it up; if it is tails, you send it down; you get some two point processes.

Let us call this $N_1(t)$. This process is called $N_1(t)$. This process is called $N_2(t)$. Now, we want to characterise the process that comes after splitting. So, you want to characterise $N_1(t)$ and $N_2(t)$. So, the original process is Poisson of rate λ. You want to know what property does $N_1(t)$ and $N_2(t)$ have? Remarkably, $N_1(t)$ and $N_2(t)$ are also Poisson processes. So, that is the main result. So, in the setting above, number 1, $N_1(t)$ is a PP of rate λp.

"Professor - student conversation starts" So, exactly what is your question? You said that if the coin toss is not; so, you are looking at; what you are saying; see, the; it may. So, each time you are throwing with, the coin toss is, coin is thrown, and it shows up heads with probability p, and it is; see, that only characterises the marginal of that coin toss. I am saying that the joint distribution of all these coin tosses is independent of; I mean, they are all independent and independent of the X_i's. That is what I am saying.

Yeah, correct. The coin toss does not look at the process. No, I mean; so, what do you; your question is, I guess, what if you are likely to show up heads more likely if your interarrival times are large or something like that; then this will not hold; is that your question? It will not hold. Yeah, they will not be Poisson. So, what I mean is, if the coin toss showing up heads is somehow looking at the past inter arrival times, then this is not true.

I think that is her question. That is not true. Then this property will not come up. This is an IID Bernoulli split, independent identically distributed coin toss. **"Professor - student conversation ends"** And $N_2(t)$ is a Poisson process of rate $\lambda(1 - p)$. So, you split a Poisson process in this IID Bernoulli manner, you get two Poisson processes of rate λp and $\lambda(1 - p)$. That is, I think, it is remarkable in itself.

What is even more remarkable is that the split processes turn out to be independent. See, they are coming from the same Poisson process. So, furthermore, this is the most remarkable thing; $N_1(t)$ and $N_2(t)$ are independent Poisson processes. I think that the second thing is even more remarkable. See, the first thing; see, you are getting λ arrivals per second, roughly, and you are sending a fraction p of them up.

So, the up process should have rate λp; that much is fairly straightforward. And the down process should have rate $\lambda(1 - p)$. But what this theorem is saying is that, they are separately each Poisson processes. And furthermore, these two processes are independent, which is absolutely, in my opinion, absolutely surprising; because they are coming from the same process, but it turns out that they are independent.

So, here is my point; if I split; see, if I give you two independent Poisson processes of rates λp and $\lambda(1 - p)$. I just give you two independent processes, versus, I give you, I take a λ process, Poisson process and split and give you the two splits. You cannot tell the difference. Say, if there are two Poisson processes coming from this IID Bernoulli split, there is no way to tell them apart from two processes generated independently.

Consider; so, this is pretty amazing. Suppose you have these radioactive particles coming at you, let us say from a sample. Suppose I split them p and $(1 - p)$ and I give you only these two processes, they will behave as though they are coming from two separate samples of the corresponding rates, although they are coming from the same; so, this is a very remarkable property. This is very highly non-trivial. Again, you can prove this in any number of ways, but I think the simplest is again look at this λδ business.

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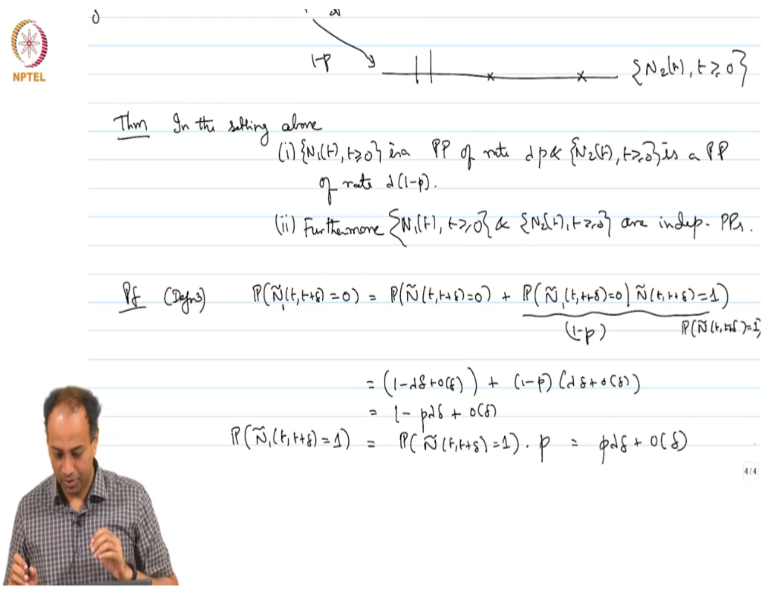

So, we can use this definition 3. So, you look at a $(t,\ t+\ \delta)$ times. I am going to look at the $P(\widetilde{N}_1(t,\ t+\ \delta)\ =\ 0)$, meaning that, in some interval $(t,\ t+\ \delta)$, there is no arrival in the up process. So, this can happen in two; there are two possibilities. I am looking at this $(t,\ t+\ \delta)$, and I want no arrivals in it. So, maybe that the original process had no arrivals, in which case, of course, there will not be any arrivals; or, the original process had an arrival, but it was sent down.

So, this is split into two disjoint events, one corresponding to $\widetilde{N}\ (t,\ t+\ \delta)\ =\ 0$ or $\widetilde{N}_1(t,\ t+\ \delta)\ =\ 0$ given $\widetilde{N}\ (t,\ t+\ \delta)\ =\ 1$, and it went down, which has probability 1 - p. I am just using; **"Professor - student conversation starts"** See, I am ignoring; so, again, this is true in the limit when δ is very small, because there could be more than two arrivals and all that.

So, this is not really equal, this is approximately equal and becomes more and more true. See, this is the problem with this λδ approach; there is some approximation involved, but it is sometimes the most transparent way to see things. So, this is not really equal to, because there could be potentially two arrivals in the original Poisson and both of them went down. **"Professor - student conversation ends"**

What I wanted to say is that, this guy is 1 - p times; sorry, I made a mistake; I am just writing total probability here, right?

$$P(\widetilde{N}_1(t,\, t+\, \delta) = 0) \; = \; P(\widetilde{N}\, (t,\, t+\, \delta) = 0)$$
$$+ \; P(\widetilde{N}_1(t,\, t+\, \delta) = 0 |\, \widetilde{N}\, (t,\, t+\, \delta) = 1).\mathrm{P}(\widetilde{N}\, (t,\, t+\, \delta) = 1)$$

Sorry, I made a mistake. So, now, I think I have corrected it. So, I am writing; first, there is no arrival at all in the first process, or there is an arrival and that guy went down. So, this is what?

$$P(\widetilde{N}_1(t,\, t+\, \delta) = 0) \; = (1 - \lambda\delta + o(\delta)) \; + \; (1 - p)(\lambda\delta + o(\delta))$$

So, what does that work out to be? That is equal to; see, this λδ will cancel that λδ. So, you will get $1 \; - \; p\lambda\delta \; + o(\delta)$; $o(\delta) \; + \; (1 - p)o(\delta) \; = \; o(\delta)$. Now, likewise, you can write, what is the $\mathrm{P}(\widetilde{N}_1(t,\, t+\, \delta) = 1)$? This is equal to the $\mathrm{P}(\widetilde{N}\, (t,\, t+\, \delta) = 1)$, and it was sent up.

So, basically, I should write $P(\widetilde{N}_1(t,\, t+\, \delta) = 1 \,|\, \widetilde{N}\, (t,\, t+\, \delta) = 1)$, which is simply $\mathrm{P}(\widetilde{N}\, (t,\, t+\, \delta) = 1)$ times p. So, this is equal to $p\lambda\delta \; + o(\delta)$. Got it? So, you are getting this Poisson property, at least the incrementals are right. Now, you have to prove SIP and IIP. Now, SIP and IIP will come. You have SIP and IIP for the original process, and these Bernoulli coin tosses are independent in any of these little slots.

If at all there is an arrival, whether you send it up or down, is independent of what happened in the previous slots. From this, you will get, you can prove SIP and IIP for the up process and the down process separately. **"Professor - student conversation starts"** Yeah, exactly. You can do it. I am doing a very rough calculation. See; it would not make a difference. Yeah, you are right. So, see, this is not really a proof.

If you were to write a textbook, and if you give this as a proof, people will not accept it as a rigorous proof. I am just indicating how it goes. You are right. So, this is not strictly true, this is approximately true, and you send δ to 0, it becomes, the approximation gets better and better. **"Professor - student conversation ends"** See, the other remark I wanted to make is that this λδ approach is not enough to prove statement (ii).

See, we only proved that the first process is Poisson. If you prove SIP, IIP, you will get first Poisson with rate $p\lambda$. You can similarly prove that the down process is Poisson with rate $(1-p)\lambda$. That you can do; but independence across these two processes, you cannot prove. This is a much deeper property. And I will continue on this tomorrow, because it is a pretty subtle property. I will tell you just one sentence and finish this lecture.

After splitting, the independence of these two processes comes from this very innocuous term of $o(\delta)$. Try digesting this now. And we will do this next class. See, if you had no $o(\delta)$ term. So, namely, I tell you that there is one arrival with probability λδ, and 0 arrivals with probability $1-\lambda\delta$, and that is it, no $o(\delta)$. It is like, this is really like a Bernoulli; 1 or 0. I am totally ruling out multiple arrivals in a time slot.

Then, in that case, if I know that an arrival went up, then I know that, in that micro slot, there cannot be an arrival in the down process. So, in that case, are these two processes independent? They cannot be independent, because, if I know that there is an arrival in the up process, I know for sure that there is no arrival in the down process. But, in the Poisson process case, that is not actually true; they are independent, the theorem says they are independent.

Why? Because there is ever so small a probability that there are two arrivals. So, it is not completely ruled out. So, this $o(\delta)$ which has just been sort of a nuisance for so long, it is actually very important, that is what enables this independence. Without that $o(\delta)$; if that

$o(\delta)$ were precisely 0, then you cannot get independence, as I just argued. So, do not ignore that $o(\delta)$.

That is where I want to stop this lecture. Tomorrow, I will tell you how this independence is actually shown. It is a non-trivial property; and you cannot use the $\lambda\delta$ technique, you have to go through one of the other two definitions.