Stochastic Modeling and the Theory of Queues
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Module - 2
Lecture - 11
Merging of Poisson Processes - Part 1

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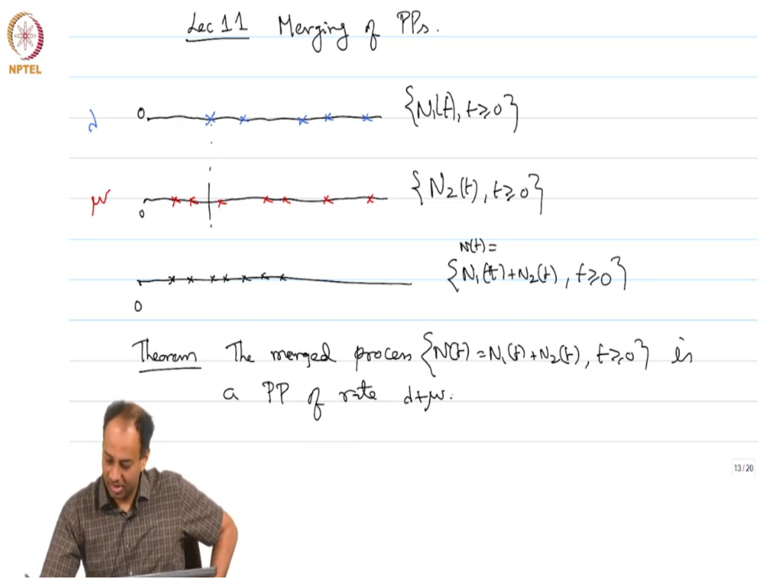

So, now, let us discuss merging of Poisson processes. Here, the setting is as follows: You have two poisson processes which are independent. You have a, let us say a Poisson process of rate λ, which, I will call a blue process. And I have another Poisson process of rate μ, which is a red process. If you do not have two colours of pens, you can just write crosses and dots or something like that.

Since I am writing on a Wacom, I can use different colours, so it is easier. Now, I said these two processes are independent, what does that really mean? These two are Poisson processes; one is a λ process, one is a μ process, meaning that, for the blue process, the interarrival times are IID exponential parameter λ. For the μ process, the interarrival times are IID exponential parameter μ.

Now, these two processes are independent, meaning that, if you look at the sequence of the random variables of interarrival times or arrival epochs in the blue process, they are independent of the sequence of arrival times or epochs or $N_1(t)$ or $N_2(t)$ of the other process. So, let us call this $N_1(t)$. This counting process is $\{N_1(t),\ t\ \geq 0\}$; this counting process is $\{N_2(t),\ t\ \geq 0\}$. Merging means, I want to merge the two.

So, I want to look at; time starts at 0, of course. I want to merge, but I am going to be colour-blind, I can not see any colours. So, I will only see that there is some arrival. So, this guy will come here and this; you see what I mean, right? This guy will come here, this guy will come here, and so on; there will be an arrival here. You see what I mean, right? So, in the merged process, there is an arrival at a particular time, if there is an arrival in either of the processes.

So, this is just the process. So, this will be the process $N_1(t)\ +\ N_2(t)$. So, the question is; so, this new process, the merged process $N\ (t)$ is just $\{N_1(t)+N_2(t),\ t \geq 0\}$. So, at any given time t, the total number of arrivals is the sum of the number of arrivals in the two processes. Now, his question was, in the merged process, how do you know that two arrivals occurring together has zero probability?

That is what you have to prove. Again, you go back; this is a good question; you go back to the δ view of the world. In a δ slot, what is the probability of having two or more arrivals in any given slot? See, it has to be $\lambda\delta\ +\ o(\delta)$ times $\mu\delta\ +\ o(\delta)$ plus other $o(\delta)$terms, which will become $\lambda\mu\delta^2$already, which is $o(\delta)$, it is like 0, very small.

So, it is a good question; I think you should probably do this as homework. Argue that, in the merged process, you cannot have two arrivals; having two arrivals in a very small time slot has $o(\delta)$ probability. Now, that is already, in some sense, he is already getting ahead of where I want to go. The remarkable thing is that the merged process $\{N(t)\ =\ N_1(t)\ +\ N_2(t),\ t \geq 0\}$is a Poisson process of rate $\lambda\ +\ \mu$.

This is a remarkable property. If you merge two independent Poisson processes of different rates, the rates can be; λ and μ do not have to be equal, they can, one can have a much higher rate, one can have a smaller rate; all that is okay. If the merged process has a rate $\lambda + \mu$, and it is a Poisson process. **"Professor - student conversation starts"** No, no. So, if I merge two processes, I am Poisson; so, if I merge fourteen processes of different rate, I am still Poisson. That follows, right? I do not have to; because, that if it merged the third process, I can merge with $N_1(t) + N_2(t)$, which is already a Poisson process. See, the processes have to be independent, that is all. **"Professor - student conversation ends"** So, likewise, you can argue that if you have some finite k number of Poisson processes which are independent, of rates $\lambda_1 \ldots \lambda_k$, you merge all of them, you will get a Poisson process of rate $\lambda_1 + \lambda_2 + \ldots + \lambda_k$.

"Professor - student conversation starts" No, this is the theorem I want to prove. No, this is a theorem I am claiming, I have not proved it. No. So, how do you prove this theorem is the question. **"Professor - student conversation ends"** So, presumably you can use any of the three definitions. See, now, all these definitions are equivalent, so, you can use any one of them. In this case, perhaps the easiest one to use is the δ micro slots view of the world. So, you can use definition three.

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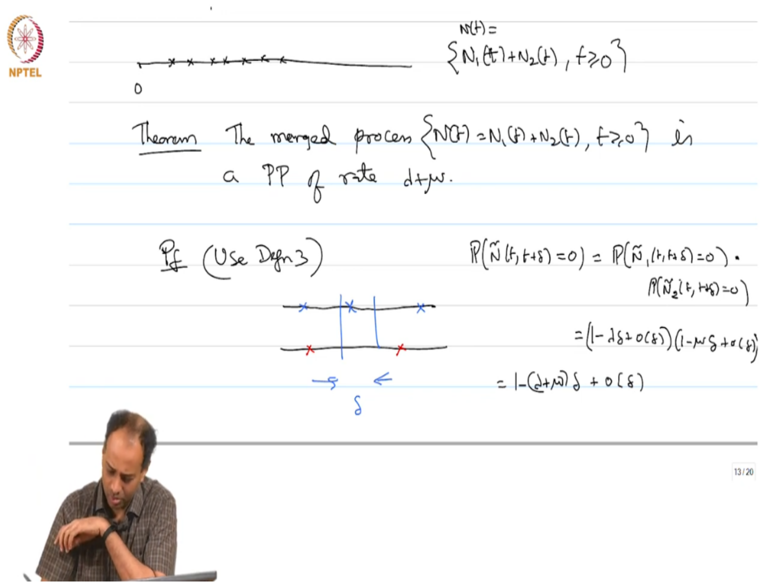

Proof: Use definition three, which is the easiest way to prove this. So, basically, you have a δ time slot. So, let us say this is the δ time slot; and this is like δ; and this is the blue process;

and this is the red process. Now, what is the $P(\widetilde{N}(t,\ t+\ \delta)\ =\ 0)$? Meaning that, in this δ time interval, there is no arrival in the merged process.

This is the same as the event that the first process has no arrivals and the second process has no arrivals. Now, the first process and second process are independent. So, you can multiply. So, this will be the $P(\widetilde{N}_1(t,\ t+\ \delta)\ =\ 0)\,.P(\widetilde{N}_2(t,\ t+\ \delta)\ =\ 0)$. This is nothing but $(1\ -\ \lambda\delta\ +\ o(\delta))\,.\,(1\ -\ \mu\delta\ +\ o(\delta))$.

That is because; I am using definition three; so, if you just write this out, what happens? This becomes $1\ -\ (\lambda\ +\ \mu)\ \delta\ +\ o(\delta)$. All this can be; you can eat all of that into another$o(\delta)$ term. Correct? With me?

This is because of; see, I am looking at definition 3, which is this micro slot view, and I am using the independence of the two processes. So, I can multiply the probability of having no arrivals. Likewise, I can look at $P(\widetilde{N}(t,\ t+\ \delta)\ =\ 1)$, which means that you have to have exactly one arrival. You should have one arrival from the first process and no arrivals from the second process, or you have none from the other way around, which you can write out $\lambda\delta.\,(1\ -\ \mu\delta\ +\ o(\delta))\ +\ \mu\delta.\,(1\ -\ \lambda\delta\ +\ o(\delta))$.

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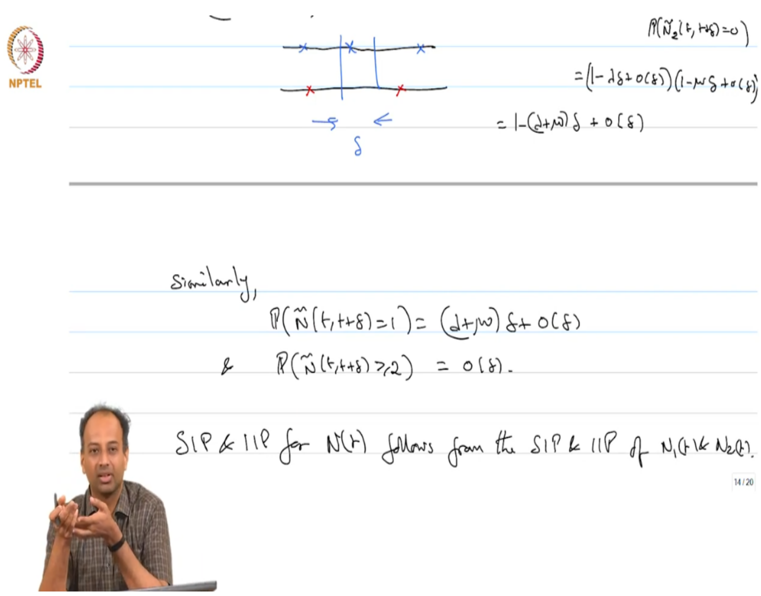

Likewise, we can prove that $P(\widetilde{N}(t,\ t+\ \delta)=1)\ =\ (\lambda+\ \mu)\delta\ +\ o(\delta)$, similar sort of an argument. And $P(\widetilde{N}(t,\ t+\ \delta)\geq 2)$. You must have one arrival in the first process and one arrival in the second process, which has probability $\lambda\mu\delta^2$, and plus there may be 0, 2 or something like that, which also has a $o(\delta)$ probability.

So, this, you can again show is of probably $o(\delta)$. **"Professor - student conversation starts"** I think this answers your question. Probably of having two or more arrivals in the composite merged process, is $o(\delta)$. So, again, to answer your question. You do not need it. I have just proved it. **"Professor - student conversation ends"** So, I have gotten the; in these micro slots, I have the right distribution, and I need to prove that the merged process has;

"Professor - student conversation starts" No, the rate is coming out, $\lambda+\ \mu$ has already come out. **"Professor - student conversation ends"** I have to prove that it still has SIP and IIP. See, I know that the individual processes have SIP and IIP. So, SIP and IIP for $N(t)$ follows from the SIP and IIP of $N_1(t)$ and $N_2(t)$. What do I mean? So, if I want to prove that increments in this merged process are independent; if you look at two different intervals; I want to prove independence; it is true because the constituent processes have independence, and now the two processes are themselves independent.