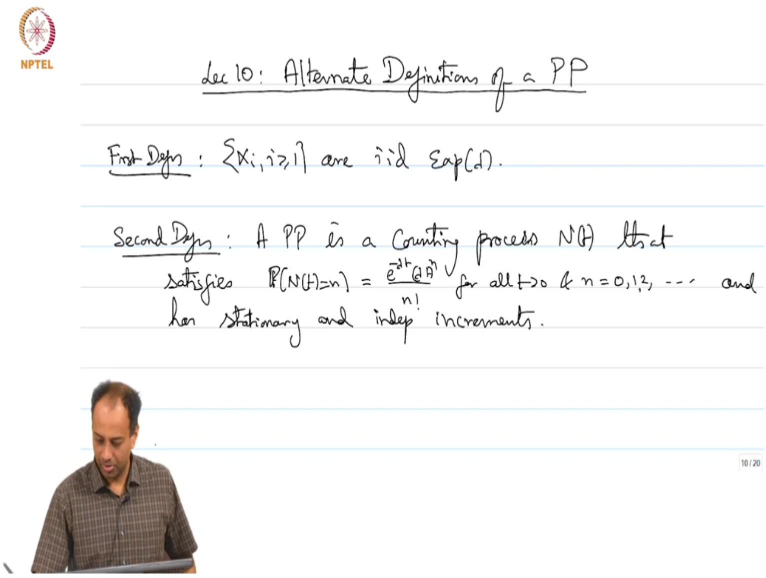
Stochastic Modeling and the Theory of Queues
Prof. Krishna Jagannathan
Department of Electrical Engineering
Indian Institute of Technology - Madras

Module - 2
Lecture - 10
Alternate Definitions of a Poisson Process

(Refer Slide Time: 00:14)



Good morning. Yesterday we derived the distributions of the arrival epochs S_n and the PMF of $N(t)$. Today, we will discuss alternate definitions of a Poisson process. So, the definition we have seen of a Poisson process is that the inter-arrival times are independent and identically distributed exponential random variables. There are two other definitions which are equivalent. So, there are totally three equivalent definitions. So, that is what we will see today.

The first definition is something we have seen already. This is basically that $\{X_i,\ i \geq 1\}$ are IID exponential random variables of some parameter λ. The second definition says following: Poisson process is a counting process $\{N(t),\ t \geq 0\}$ that satisfies $P(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!},\ n = 0, 1, 2...$ and has independent increments. So, this is the second definition. It says that the Poisson process is a counting process that satisfies

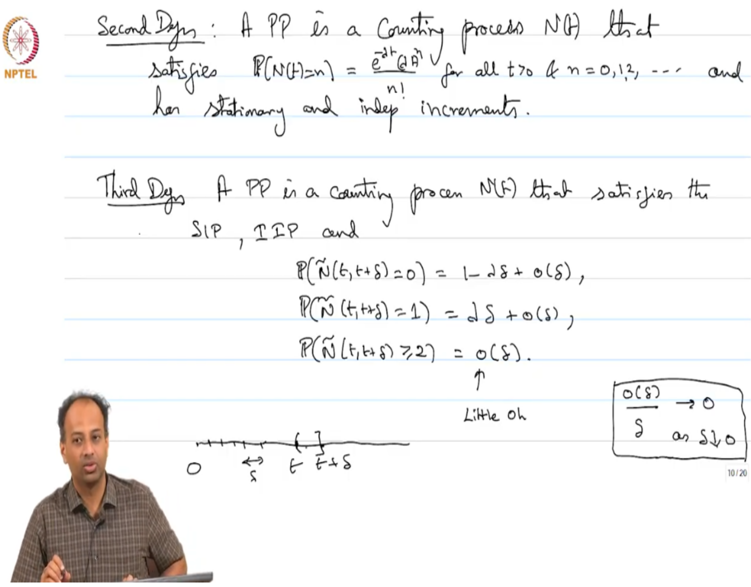
$P(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n = 0, 1, 2...$ and for all t > 0, and it has to have SIP and IIP, stationary increment property and independent increment property.

This is the second definition. Now, so, please note; the main thing to note is that, just saying that $N(t)$ satisfies the Poisson PMF for every t and every n is not enough to specify a Poisson process. If I just tell you that a counting process has Poisson PMF for every t and every n, $N(t)$ has a Poisson PMF, it is not enough; there are other processes which have that property. So, the Poisson process has the Poisson PMF and has SIP and IIP.

Then, that is a unique specification; it turns out. So, we have already shown that the first definition implies the second definition. Because, we defined the Poisson process as a, this IID exponential interarrival times. Then we derived the Poisson PMF from it; we also derived the IIP and SIP from it. So, the process as specified by the first definition will satisfy the second definition.

See, whenever you say that; see, when you define the same thing in multiple ways, you should show that they are equivalent, meaning that one implies the other. So, if you give two definitions, I should show that the first definition implies the second definition and the second definition implies the first definition. So, the first definition implies the second definition, we have already shown. Second definition implies the first definition, we have to show. That is pending. There is also a third definition.

(Refer Slide Time: 04:13)



A Poisson process is a counting process $\{N(t),\ t\ \geq\ 0\}$ that satisfies the stationary increment property and independent increment property and the following:

$$P(\widetilde{N}(t,\ t\ +\ \delta)\ =\ 0)\ =\ 1\ -\ \lambda\delta\ +\ o(\delta)$$

$$P(\widetilde{N}(t,\ t\ +\ \delta)\ =\ 1)\ =\ \lambda\delta\ +\ o(\delta)$$

$$P(\widetilde{N}(t,\ t\ +\ \delta)\ \geq\ 2)\ =\ o(\delta)$$

So, the third definition here looks at increments in a small interval. So, the second definition says, you have Poisson PMF and SIP and IIP. Third definition says, SIP, IIP and a certain incremental specification, over what does the process do in a small interval δ. So, if you look at; this is time 0; you look at some time t and time $t\ +\ \delta$. So, you look at the number of arrivals that occur in $(t, t\ +\ \delta)$.

It says, this condition says that you essentially have 0 arrivals with probability $1\ -\ \lambda\delta$; you have 1 arrival with probability $\lambda\delta$. And this $o(\delta)$ is a very small term. Are you familiar with the o notation? This o is not O. This guy is a little o. So, a $o(\delta)$ basically means that; so, a function is $o(\delta)$;$\frac{o(\delta)}{\delta} \rightarrow 0$ as $\delta\ \rightarrow 0$.

So, a $o(\delta)$ is a term that contains δ^2 or δ^3 and so on. So, you might have seen this notation before. If you have not, essentially, this is all that it means. So, you have 0 arrivals with probability $1 - \lambda\delta + o(\delta)$; and 1 arrival with $\lambda\delta + o(\delta)$. And more than, 2 or more arrivals is very unlikely, it is a very small probability, $o(\delta)$.

Now, so what it really says is that, if you look at these sort of δ micro slots, if you derive time into these little δ micro slots; see, the stationary increment property implies that it does not matter what t is. So, in any δ interval, this t is irrelevant. In any δ interval, there is, this is satisfied. And independent implement property means that, if you consider two of these δ micro slots, the number of arrivals in each of them is, they are independent random variables.

And furthermore, what this is saying is that, if you consider these δ micro slots, the Poisson process essentially looks like a Bernoulli process. What is a Bernoulli process? See, a Bernoulli process, it is a discrete time process; so, time goes, not like, not continuously, but $t = 0$, $t = 1$, $t = 2$, and so on. And at each of these times, you have a Bernoulli arrival, meaning that you have an arrival with probability p, or no arrival with probability $1 - p$. You know what a Bernoulli random variable is, right?

So, Bernoulli process means, it is a discrete time process, and each time slot, you have an arrival with probability p, no arrival with probability $1 - p$, and they are independent across these discrete time slots. That is what a Bernoulli process is. Now, what we are saying is that, the Poisson process is very much like a Bernoulli process, if you look at these micro slots.

So, in each of these δ intervals, there is an arrival with probability λδ, and no arrival with probability $1 - \lambda\delta$. And independent increment property is satisfied; so, these Bernoullis are independent. And of course, the t does not matter; stationary increment property. **"Professor - student conversation starts"** Which one? This definition? It is not a discrete IID, it is a continuous time process.

No, all I am saying is that, if you look at these small time slots of δ, in these little time slots; assume that δ is very small and it is going to 0. In this very little δ time slots, in each of these, it behaves like an independent, it looks like a IID Bernoulli random variable, whose arrival probability is $\lambda\delta$. It is a continuous time process. It is still a continuous time process. But if δ is small enough, it looks like a Bernoulli process.

That is all that we are saying. **"Professor - student conversation ends"** So, it is like a independent increment Bernoulli process as these time slots are becoming smaller and smaller. So, perhaps this is an easier way to understand it. You consider a Bernoulli process, where the time intervals are not $0,\ 1,\ 2$, but $0,\ \delta,\ 2\delta$, and so on. They are δ time slots, and in each of these time slots, there is an arrival with probability $\lambda\delta$.

Now, this arrival probability is not p, it is not a fixed p, but it is proportional to the width of the interval. And no arrival probability is $1 - \lambda\delta$. Now, in some sense; in fact, your book does this in a more rigorous way. If you send $\delta \rightarrow 0$, then this Bernoulli process basically becomes a Poisson process. I do not want to get into the technicalities of it, because I have not really defined convergence of stochastic processes and all that.

It gets a little more technical. But intuitively, this is a very reasonable way to think about it. What is going on is a Bernoulli process in very small time slots, where the arrival probability in every time slot is proportional to the width of the interval. It is not a fixed p, but it is $\lambda\delta$. But there is ever so small a probability of having 2 arrivals; so, in that regard, it is not quite a Bernoulli process.

Bernoulli process means, you have 1 arrival or no arrivals. But this Poisson process gives you an ever so small probability, $o(\delta)$, of having 2 or more arrivals. This is very small, but not 0. So, this $o(\delta)$; so, it is a little innocuous thing, but I will point out later that it is actually very important to have this. If you did not have this at all, if $o(\delta)$ that precisely 0, then it is not a Poisson process.

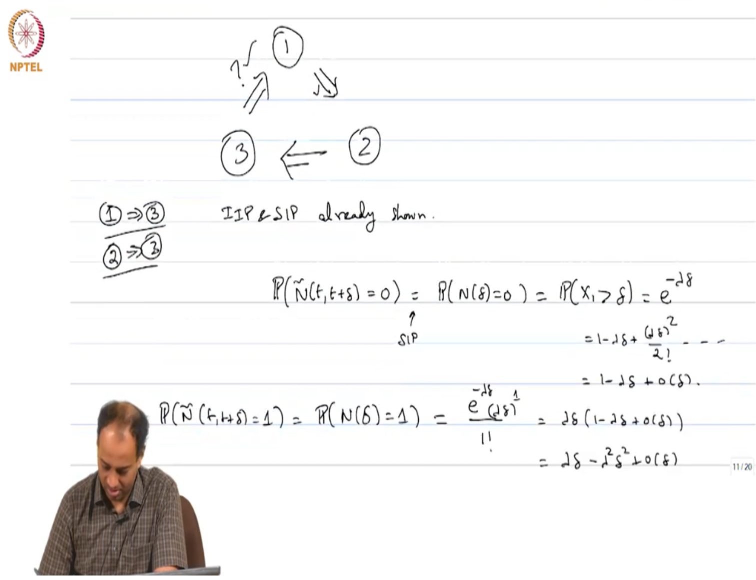
Some of these fundamental properties of the Poisson process will not play out if you have a probability of 2 or more arrivals if it were precisely 0. Then some of these; it is not a Poisson process, and some of the important properties of the Poisson process will not come out. So,

$o(\delta)$ is very small, but not really 0. And this matters. And how it matters, you will not see now, but I will point out later.

It is not t goes to 0; I have written it; it is t > 0. So, this guy; so you are talking about this, right? Maybe I will write this properly. For all; so, I am just saying that there is a Poisson PMF for all t > 0. So, we have given 3 different definitions of the Poisson process. Now, as I said, if you have, if you pretend to define the same thing in 3 different ways, it is better that you show; you have to show that they are all equivalent, which means that you have to show 1 ⇒ 2, 2 ⇒ 1, 1 ⇒ 3, 3 ⇒ 1, 2 ⇒ 3, 3 ⇒ 1.

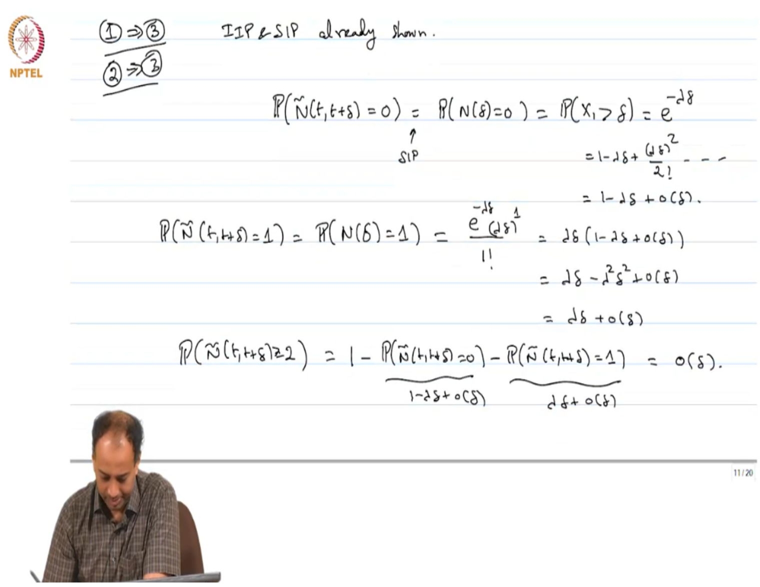
So, that is how many equivalences to show? 6 equivalences you have to show. Or you can show that 1 ⇒ 2 ⇒ 3 ⇒ 1. That is also okay; that is enough.

(Refer Slide Time: 13:52)



So, we have to show that; so, I am just putting a picture. This is definition 1, definition 2, definition 3. This implication, you already know. Because you derived the Poisson PMF and SIP and IIP from the exponential definition. Is 1 ⇒ 3 easy? Let us, perhaps you should do 1⇒ 3.

(Refer Slide Time: 14:23)



So, let us prove that 1 ⇒ 3 for example. So, you have independent exponential interarrival times. You want to prove that $N(t,\ t+\ \delta)$ satisfies this property. So, see, a process satisfying 1, meaning the IID exponential times, satisfies IIP and SIP. That I have already shown. So, the only thing left to show is that; so, IIP and SIP already shown, through that big theorem we proved.

Now, I just have to show that $P(\widetilde{N}(t,\ t+\ \delta))$ satisfies that increment. So, you have to look at, let us say, $P(\widetilde{N}(t,\ t+\ \delta)\ =0)$; is equal to; of course, this is equal to $P(N(\delta)\ =\ 0)$. Why? This is because of SIP, because I have already shown SIP. So, in $(0,\delta]$, there should be no arrivals. What is the probability that there is no arrival in $(0,\delta]$?

This is same as the $P(X_1>\ \delta)$. I proved this, no? $\{N(t)\ =\ 0\}\ =\ \{X_1>t\}$. Now, this is equal to what? $P(X_1>\ \delta)\ =\ e^{-\lambda\delta}$. Definition 1 says that X_1 is exponentially distributed. So, this is nothing but,

$e^{-\lambda\delta}=\ 1\ -\ \lambda\delta\ +\ \frac{(\lambda\delta)^2}{2!}.....$

So, if you write out this in Taylor expansion, you will get $1\ -\ \lambda\delta\ +\ \frac{(\lambda\delta)^2}{2!}....$

We will get δ^2 terms, δ^3 terms and all that. $e^{-\lambda\delta}$ you expand in Taylor. And this is just $1 - \lambda\delta + o(\delta)$. Next, you can show $P(\tilde{N}(t, t + \delta) = 1) = P(N(\delta) = 1)$. So, $N(\delta) = 1$, in $(0, \delta]$, you should have 1 arrival, which means that $X_1 \leq \delta$. And you know, there is another way you can approach this.

We have already shown that there is an easier way to approach this, in the sense that, we have already shown 1 ⇒ 2. So, the probability that $P(N(\delta) = 1)$ is; I know the Poisson PMF. I am cheating a little bit, because I am not going directly from 1 to 3. You can do it, but I am saying, you can; you have already proven 2, so, you can use 2. This is logically correct. 1 ⇒ 2, I have done. So, something I know from 2, I can use.

So, I am cheating a little bit. So, this is from definition 2, but I have already proven 1 ⇒ 2. So, this you can write it as; so,

$$P(N(\delta) = 1) = \frac{e^{-\lambda t}(\lambda t)^1}{1!} = \lambda\delta(1 - \lambda\delta + o(\delta))$$

How did I get that? $e^{-\lambda t}$ is expanded in Taylor. **"Professor - student conversation starts"** Yeah, I could have done the same thing.

I have already proven that 1 ⇒ 2. I want to prove 1 ⇒ 3. So, I am going through; essentially, what I am; what you are saying is that, what I am doing by cheating is going from 2 to 3. So, yeah, this is easy; 2 to 3 is easy. So, this is both; so, you are right; 2 to 3 is also here. So, yeah, you are right; what you are saying is right. So, I am going; so, already went from 1 to 2. So, if you want to go from 1 to 3, I can go through 2.

So, you are right, I do not disagree at all. I think what she is saying is that, $P(N(\delta) = 0)$; since I have already proven 2, I can use the Poisson PMF. I agree with it. **"Professor - student conversation ends"** So, now, this is nothing but

$$P(N(\delta) = 1) = \lambda\delta - \lambda^2\delta^2 + o(\delta) = \lambda\delta + o(\delta)$$

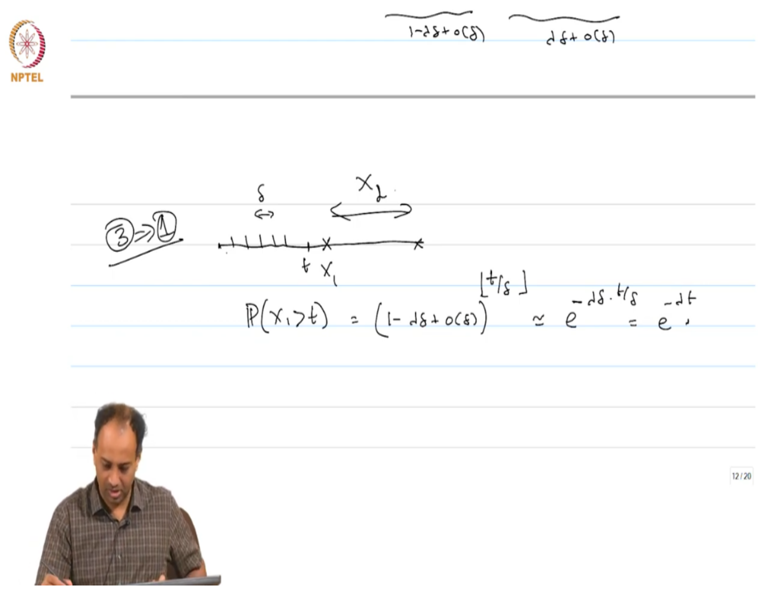
Now, this guy is another $o(\delta)$ term, because there is a δ^2 there. So, from this, it is obvious that,

$$P(\widetilde{N}(t,\ t\ +\ \delta)\ \geq\ 2)\ =1\ -P(\widetilde{N}(t,\ t\ +\ \delta)\ =0)\ -P(\widetilde{N}(t,\ t\ +\ \delta)\ =1)$$

because, it has a sum to 1, no? $\widetilde{N}(t,\ t\ +\ \delta)$ may be 0, 1 or 2 or bigger; all of this has to add to 1. Now, this guy is; these guys I already know, right? So, this guy is already $1\ -\ \lambda\delta\ +\ o(\delta)$. This guy is $\lambda\delta\ +\ o(\delta)$.

So, if I add these two, I get $o(\delta)$. So, this will be this $o(\delta)$. So, I have shown 1 $\Rightarrow$ 2 and 2 $\Rightarrow$ 3, and therefore, I have shown 1 $\Rightarrow$ 3. So, these are done. Now, maybe; so, if I do this, I am done. If I do that, I am done. So, if I take this Bernoulli view of this incremental, δ incremental view of the Poisson process and prove that I get exponential interarrival times, then I am done; independent exponential arrival times. I will just sketch it. I will let you do all the details.

(Refer Slide Time: 23:11)



So, I will just point out how it goes, 3 $\Rightarrow$ 1. So, you are looking at, let us say that is my X_1. See, definition 3 looks at a, this sort of a δ view of the world. I want to essentially prove that X_1 is exponentially distributed with parameter λ, and that the subsequent interarrival times are also exponential and independent of X_1. If these two I prove, I am done. First, let us prove that X_1 is exponentially distributed.

So, I want this. What is this? This is what I want to find out what it is. So, this means that, in this t; so, if you look at some t out here; you want the $X_1 > t$, which means, for the first t units of time, there should be no arrival at all; which means that, if you look at these micro slots, there should be no arrivals. So, there should be no arrival in this δ slot, and no arrival here, and no arrival here, and no arrival, so on.

So, the probability of not having an arrival in any of these micro slots is $1 - \lambda\delta + o(\delta)$. And how many are there? Approximately t/δ. And whether you have an arrival in 1 micro slot or do not, they are independent. Why? It is a part of definition 3, the increments are independent. So, the probability of having arrival here and not having an arrival here and not having an arrival here; these guys are all, you can multiply the probabilities, because they are independent.

So, you will have roughly something like floor(t/δ). If you want, you take something like that. So, for δ → 0, this will be like; see, this $o(\delta)$ is a very small term. So, this is like $(1 - \lambda\delta + o(\delta))^{[t/\delta]}$, which will be like $e^{-(\lambda\delta)(t/\delta)}$. That is equal to $e^{-\lambda t}$. This is the essential reasoning, but if you want to make this rigorous, you have to say less than or equal to, greater than or equal to; you can make it rigorous, but essentially, you will get $e^{-\lambda t}$.

So, X_1 is exponentially distributed, then you have to prove that; once that you got this X_1, first arrival, then the time to next arrival, let us say X_2 out here; so, this is not X_2, this is; then again, for X_2 we can make a similar argument because you know, independence. That X_2 is independent of X_1 is clear because all the micro slots are independent. And X_2 was also exponential by the same sort of logic; you can use the same argument.

You will get X_2 is also an exponential and independent of X_1. So, you can prove that these X_i's are IID exponentials, you can make this more formal. That is the crux of the reason. So, this also we can do. So, this question mark also you can dissolve. So, the equivalence between these 3 is shown. Now, one remark that I should make is that, look at; to prove that, something like that third definition which gives t, these δ increments and specifying distribution in δ increments with SIP and IIP, that it defines a unique counting process itself requires a proof.

Potentially, there may be multiple processes which have this property, that it is not the case and it uniquely defines the process itself. Proving it rigorously is a non-trivial matter, but we will not get into all that. So, third definition is useful because; you can actually take these Bernoulli sort of processes. Its small time slots, you consider Bernoulli process, and you can show that the joint distribution converges to that of a Poisson process.

There is a result in the book, which actually shows this convergence distribution of these, joint distribution of these $N(t_j)$'s. You have a Bernoulli process; shrink the times in a certain way; you can show convergence to a Poisson process; at least the convergence of joint distributions, you can show. We do not yet know what a convergence of a stochastic process is; that is a; there are many ways of, many notions of convergence, we will not even get there.

But joint distribution convergence, you can show; it is done in your book; that the small micro slot Bernoullis actually converges to a, this Poisson process limit, you can show; not the process limit, but the joint distribution; convergence in distribution, you can show. So, now, the other remark I wanted to make is that we have shown that these 3 definitions are equivalent. So, you could have started with any one of these.

Usually, most people start with the first definition, because it is the easiest to understand. But depending on what problem you are trying to solve, or what application you are looking at, the second definition or third definition may be more useful or more natural to use to solve that problem. So, for which application or which problem or which exercise; which definition you apply, depends on this particular problem you are trying to solve.

They are all equivalent, of course, you can start with any of them, but it is usually the case that one of them is the most convenient. And which is the most convenient definition to use, you get by practice; you have to solve lots of problems. So, that finishes this module on equivalent definitions.