Digital System Design
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Boolean Algebra

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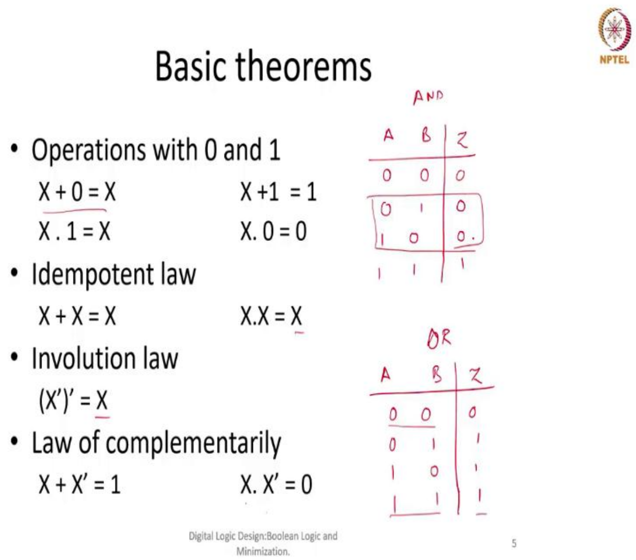

So, the basic theorems or basic properties of Boolean algebra is, these are the 4 ones. One that if I want to do an operation with 0, then the OR operations, I will quickly write the truth table of AND and OR. So, this is truth table of AND, this would help me. A, B and let us the output is Z. 0 0 0, 0 1 0, 1 0 0, 1 1 1. Now, similarly, for OR gate also if I write A, B and Z, 0 0 0, 0 1 1, 1 0 1, 1 1 1.

Now, I want, I am saying I want to do an operation X OR 0, what is the output? So, let us say this is the X, A is my X and if I am operating with 0, so if X was 0, A was 0 and I am doing an OR operation with 0, output was 0. If A was 1, I am doing OR operation with 0, again it was 1. So, that means whatever was the value of A and if I am doing an operation with the 0, I am doing an OR operation with 0, the output was also A and this is what is written over here that if I am doing an OR with a 0, then variable will remain same.

Similarly, if I am doing an operation with 1, my output is going to be 1, so I am saying either variable X OR 1, so that means whatever is the value of X, whether it is 0 or 1, my output is always going to be 1. Similarly, we can also see the operation with 1 for the AND gate. If X, whatever is the value of X, if I am ANDing it with 1, the variable will remain same. So, let us say this A is X, so I am doing an operation with 1, so if A is 0 and doing and AND with 1, output is 0. If A was 1, and I am doing an operation with 1, the output is 1 so that means this A remain same if I am doing an AND operation with 1.

So, on the other hand, if I am saying X AND 0, so whatever is the value of X, my output is always going to remain 0, so that is how we can remember these operations, these operations are very, very important or useful whenever we are doing logic minimization. Now, Idempotent law, idempotent means like if I am doing an operation with itself, what is the result? So, if I doing an OR operation of X with X means this A is also X, B is also X, both of them are same, so that means either a 0 0 output is 0, if both of them are 1, the output is 1.

So, that means, if the value of X, the output is also X, so X plus X would be X here and if I am doing an AND operation of variable with itself, the output is again the same variable. Involution law means that what would happen if I am taking invert of a variable, I am taking invert of invert of a variable, so this value this remain will bring it back. So, inverse of a inverse of variable is again the same variable. So, this is what involution law says here.

The other is law of complementary, complementarily means that if I OR X with X dash, then output is going to be 1. So, this means that if X is 0, X dash will become 1, so 0 OR 1 will become 1. Similarly, if X is 1 and then X OR will become 0, X dash will become 0, so 1 OR with 0 will become 1. And X if I AND with inverter of itself of basically NOT of itself, then the output is 0 because if this is 1, then this will become 0 or if this is 0, then this will become 1, the output is this combination which means that output is going to be 0. So, these basic theorems is going to help us.

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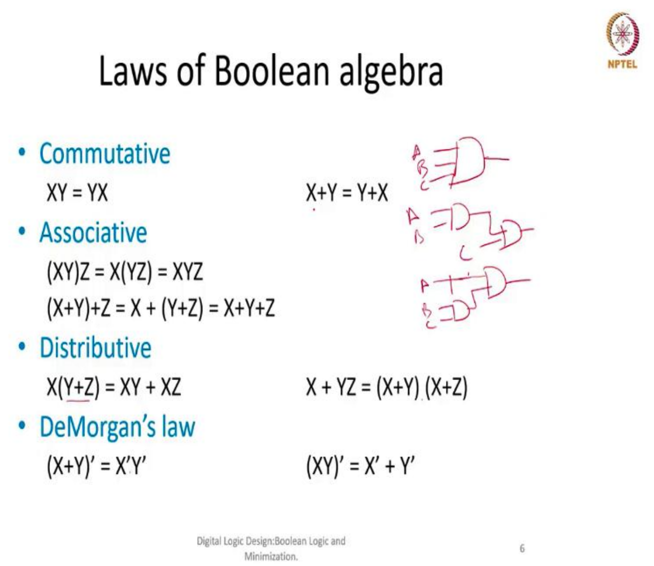

And let us consider couple of more basic theorems which are whether these operations are commutative, associative, distributive. So, yes they are commutative that means any order will not effect if we are doing an AND with X and Y or Y and X, it does not matter. Or OR operations so which essentially means that if I am writing a AND gate, it does not matter whether I am writing in which order, whether it is A B or C or C A or B or B A or C. So, all of these, the order of these variables does not matter in either AND gate or OR gate. That is what law of commutative says.

Associative means let us say I have the two operations which are in like together, so how does the order matter there, it does not matter. So, means that if I am doing an AND operation of two operations which are already ANDed, so it does not matter whether I am doing in, so this is again the similar commutative expressions which is extended to more than one variables, it does not matter what is the order, whether I am doing this AND gate like this or I am doing this AND gate like A B and doing another AND.

So, all of these things are equivalent. So, all of these 3 are equivalent whether we are creating only one AND gate or we are creating two AND gate, we are first ANDing A and B and then ANDing it with C or we are first doing B and C and then ANDing it with A. So, all of these 3 are equivalent that is what law of associative means.

Distributive is more interesting, so basically like this is a mixing of two operations. Now, Y plus Z is you are doing a OR operation and outside it is a AND, so that means if I want to AND the OR of Y and Z so that means I can also do it in a AND of X and Y, then it would be OR of X and Z. So, similarly for product also it will work if we are doing X is being ORed with a product of Y and Z or AND of Y and Z, you can also say X and Y and ANDed with X OR Z. So, this law of distribution means that if we have these operations like this, then we can reorganize these operations in the other one.

DeMorgan’s law is a inverted law that essentially says that if we are making a NOT of an expression, then all its individual variable will be inverted plus here OR would be converted into a AND. So, here also it happened like this that if we are doing a inversion of two variables like AND and AND operation of two variables, then all individual variables would be inverted and this AND would be changed to an OR gate. So, you can ask here one question that why we are representing all of these in a two different tables like you see a good similarity here, so similarity that whatever we are writing here is also written very similar way in this, the operation with OR.

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Duality principle

- Element 0 is identity element for + operator
- Element 1 is identity element for . Operator
- Duality principle:
  – Law of Boolean algebra remain same if operators and identity elements are interchanged
- DeMorgan's duality
  – A boolean expression remain same if all variable, operators are replaced with its dual

Basic theorems

- Operations with 0 and 1

  $X + 0 = X$ $\qquad$ $X + 1 = 1$

  $X . 1 = X$ $\qquad$ $X. 0 = 0$
- Idempotent law

  $X + X = X$ $\qquad$ $X.X = X$
- Involution law

  $(X')' = X$
- Law of complementarily

  $X + X' = 1$ $\qquad$ $X. X' = 0$

AND

A	B	Z
0	0	0
0	1	0
1	0	0
1	1	1

OR

A	B	Z
0	0	0
0	1	1
1	0	1
1	1	1

Laws of Boolean algebra

- Commutative

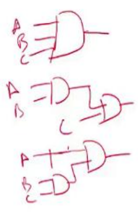
  $XY = YX$ $\qquad$ $X+Y = Y+X$
- Associative

  $(XY)Z = X(YZ) = XYZ$

  $(X+Y)+Z = X + (Y+Z) = X+Y+Z$
- Distributive

  $X(Y+Z) = XY + XZ$ $\qquad$ $X + YZ = (X+Y) (X+Z)$
- DeMorgan's law

  $(X+Y)' = X'Y'$ $\qquad$ $(XY)' = X' + Y'$



So, why it is happening because in Boolean algebra there is a interesting principle which is called duality principle. In duality principle, so before explaining this duality principle, let us understand one more fact that when we started explaining this concept of 0s and 1, when we are seeing these 0s and 1s are essentially two different states of a matter. So, there were various examples given like in a magnetic disk either it is magnetic or non-magnetic or let us say spin of a electron whether it is clockwise or anticlockwise.

Or similarly, let us say at one place if there is a light or there is no light, so similarly we can see that everywhere we see two different states, one of state we are marking as 0, another state we are marking as 1. So, which state we should mark as 0, which state we should mark as 1, does it have any kind of a impact? This is what this duality principle says. This duality principle says that we can replace wherever there is a 0 in a Boolean expression, we can go ahead and replace it with 1.

But along with that, we also have to replace plus means OR gate with a AND gate because 0 is a identity element for OR gate and 1 is a identity element for an AND gate. So, if we replace all of them, then the function would remain same, the expression would value the same. So, you can see some of these expressions if we see quickly here, if we are ANDing, so if we are saying that 0 is converted into 1, 0 is converted into 1, and plus is converted into dot, the expression is same.

Here also the plus is converted into dot, and 1 is replaced with 0, 1 is replaced with 0, expression is correct and same. Here also plus was replaced with dot means OR was replaced with AND, there was no 0 and 1, so we need not to do anything here. Here also, so we have replaced this OR with a AND gate and 1 with 0, so it worked. This is how it was working, so we can see here also like wherever we were applying we just need to replace, here there was a AND gate, this AND was replaced with the OR and this plus means OR was replaced with a AND.

So, and then this expression X dot Y will become X plus Y, X dot Z will become X plus Z. So, this duality principle says that the expression would remain same like but we have to do it on both the side. The other interesting aspect of our duality principle is that let us say this would be very much exploited in DeMorgan's law. So, in DeMorgan's law whenever we would like to invert a particular variable, so let us say we want to invert a function F, then what we need to do is inside we can invert all of its variables and replace AND with OR and replace OR with AND.

So, we will take couple of examples if not today we will take them tomorrow, so we will see that how this duality principle can help us in minimization or in looking things in a different way. So, one of the agenda of this understanding Boolean algebra was to minimize our expressions or to simplify expressions. So, based on whatever theorems we have seen here, based on these theorems now if I would like to use them, then I can create some of my simplification theorems.

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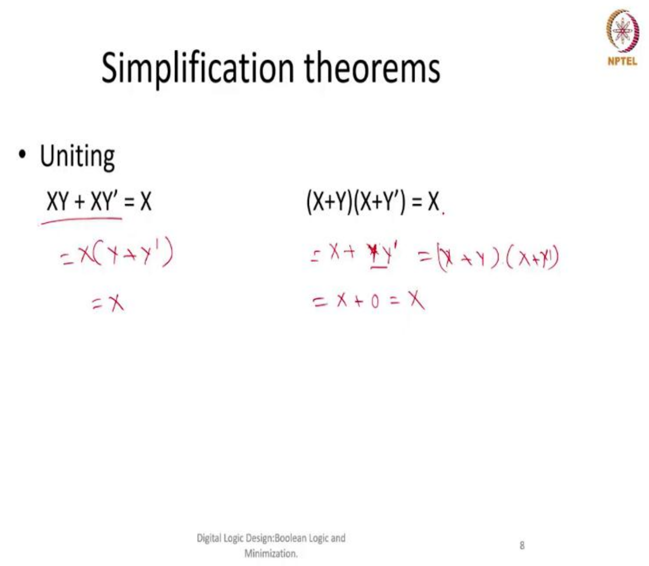

So, these simplification theorems can help me directly in simplification. So, let us say this is the expression which is given to me. Now, what I can say is because of a distributive law, I can write this as X Y plus Y dash, Y plus Y dash I know it is always going to be 1, so I can say this is equal to X. Now, there are multiple ways of seeing this. Now, again using my distributive principle, so I can also write this as X plus X, X plus Y, Y dash.

So, if you want to expand this X plus Y, Y dash using distributive law, then this is how it will become X plus Y and dot X plus Y dash. So, Y and Y dash is 0, so X plus 0 is equal to X. So, you can see that although I am writing it in a longer way but again because of simplification this is essentially is representing only X and this X plus Y X plus Y dash would actually represent X. So, this simplification could be good shortcut whenever we are simplifying our expressions.

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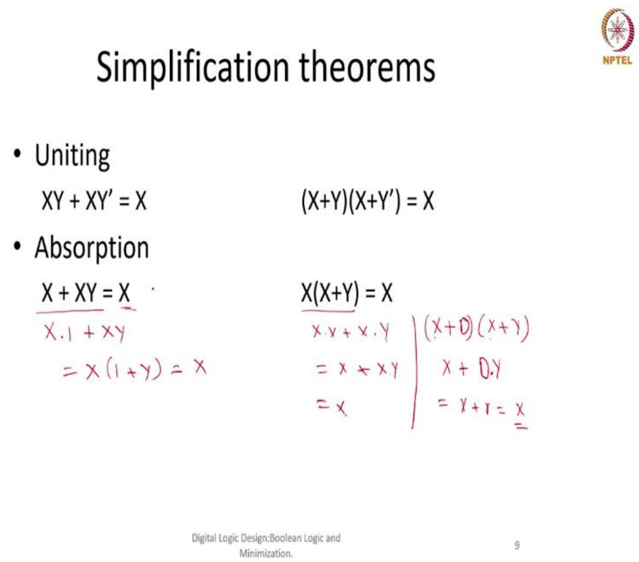

The other one absorption. So, again I can take it like this, so either I can say, X dot 1 plus XY which is equal to X 1 plus Y, 1 plus Y is equal to 1, so it can be said as X. So, this one, this also either we can go ahead and multiple it X dot X plus X dot Y which is equal to X plus XY and now we can use this principle, this will become X. In the other way we can also say that we can say that X plus 1 and X plus Y and now because it is a distributed one, so this X is here, this X is here, so we can also say X plus 1 dot Y.

1 dot Y is equal to, yeah sorry, this has to be X plus 0, so X plus 0 and now 0 dot Y is equal to 0. So, X plus 0 equal to X. this could be another way of saying this, but the overall idea is that see, this particular expression has multiple variables, multiple of logic gates would be required but after simplification you are left with only this this variable.

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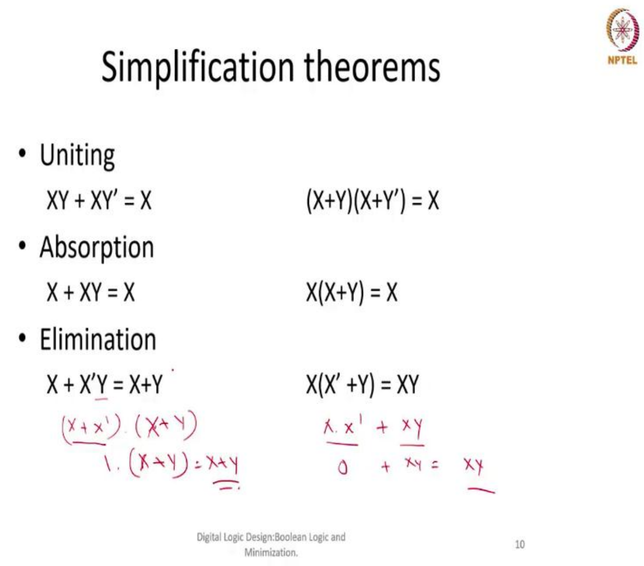

Now, let us take the other one, elimination. So, here I can say this is X plus X dash Y X plus Y. Now, I can also write it like X plus X dash and X plus Y. So, this is why, this is because of our distribution principle. Now because there is a plus here and there is a product, so I can say that I can first do a OR with this and then we will do a product and then this X and this Y could be done using an OR operation.

Now, X plus X dash is 1, 1 dot X plus Y equal to X plus Y. If we want to see this, X X dash plus Y I can simply multiply it out, I can say X dot X dash plus XY. X plus X dash is equal to 0 and X dot X dash is equal to 0 and this is XY so that is this will become XY. Now, what does these 3 simplification theorems are saying in a way? The first uniting theorem says that if I have a variable X and there is another variable Y whose complement is present, is complement is doing an AND with itself like other the variable Y as well as its complement is doing a product with doing an AND operation with these variables so then that can be united.

In absorption, we say that if there is a variable X and any other expression which has X multiplying with OR and ANDing with some other list of variables, all of them could be absorbed in X. Elimination is a interesting one because here X plus invert of X and then it is ANDed with Y so this could be this this X dash could be eliminated and we can say X it is actually equal to X plus Y. So, I can understand this could be little confusing, so we need to do couple of examples ourselves and then we would be able to understand these things better.

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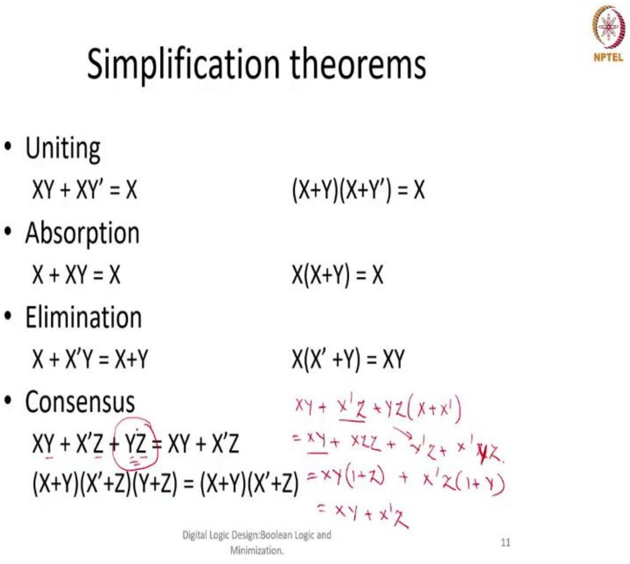

So, let us do at least, one more. So, consensus. Consensus means that you have a you have variables XY as well as X dash Z and YZ. Now, you see that this Y is here and this Z is here, so that means you have a variable, you have a variable X and there is a multiplication factor here and there is another multiplication factor with the invert of this X and that multiplication factor is also doing an OR operation. So, that means, this thing could be eliminated. So, this part could be eliminated.

How can we eliminate that? Let us take an example so let us try to prove it, we can say that XY plus X dash Z plus YZ. Now YZ we try to create another one, we say this is also multiplied with X plus X dash because X plus X dash is 1, so I am multiplying YZ with 1. This will become XY plus XYZ, I am writing it here, X dash Z plus X dash YZ, X dash this is Y and this is Z.

Now, here I can say X and Y are the common part and this is 1 plus Z and here I can say that X dash Z is a common part, 1 plus Y. Now because 1 plus Z will become 1, so this is XY plus X dash Z. So, this is what we say consensus theorem, in consensus we are eliminating whatever was the multiplication factor with X and Z if they are found like this.

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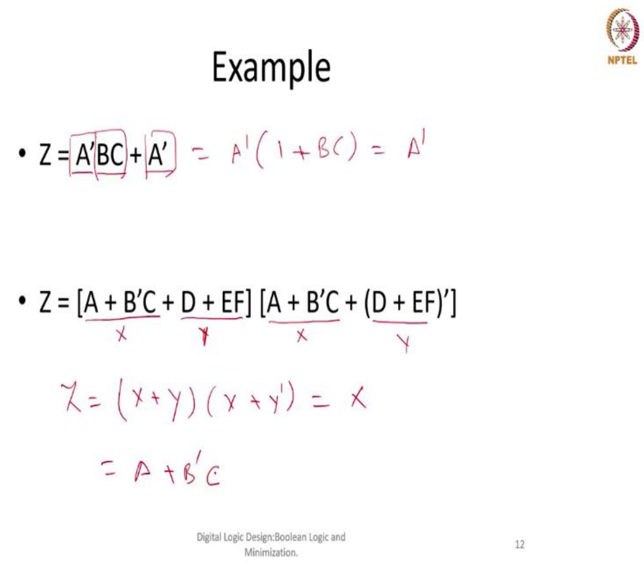

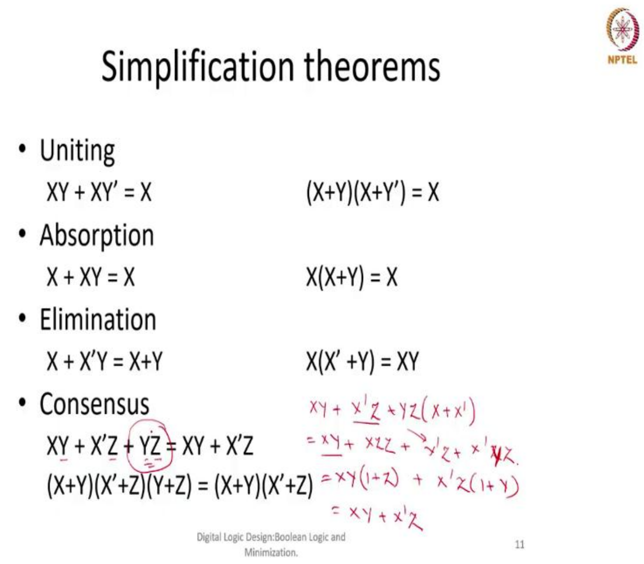

So, now these let us say couple of example to see whether we can use them and we can use them to simplify things. Now, let us take this example Z is equal to A dash BC plus A dash. What should we do? So, we see that this A dash is a common part and I can see this as one, one entity, this is can be considered as one entity. So, I can also say this as A dash, 1 plus B and C is equal to A dash. So, in other words, I can simply use this as, as this absorption theorem. So, I can call this as absorption theorem and using absorption theorem I am saying that this BC can be eliminated easily.

Now, if we have something bigger like this, so what we see that this part let me call X and this is also present in the similar form like here and D plus EF, D plus EF I see here so this I can say Y, this is also Y, so I can write this whole Z equal to X plus this is actually Y, X plus

Y and then X plus Y dash. So, I have X plus Y, I have X plus Y dash, so this will become will this will become X plus Y, X plus Y dash. So, this become X. So, this become X and my X is actually A plus B dash C.

So, this is how we can use all of these theorems to simplify. But in the end, I would summarize it like this that yes, we have this n number of theorems and these theorems can help us in minimization of Boolean expressions, but we are not always sure that what theorem to be used, what if I missed the right one? What if I am not able to use the correct theorem or can there be a systematic method?

Although we are going to use these theorems in those systematic methods also, but can there be a more systematic method? So, those systematic methods we will try to see in tomorrow's lecture, in the next lecture. By then we will try to use these theorems to simplify couple of more examples, maybe I will float a tutorial sheet along with this, along with this lecture so that you can try it out and maybe you can directly do some exercises from the textbook (())(27:00). Thank you very much.